DEFLECTIONS

AND

STATICALLY INDETERMINATE STRESSES

BY

CLARENCE W. HUDSON
M. AM. SOC. C. E.
Professor of Civil Engineering, Polytechnic Institute of Brooklyn, New York,
Consulting Engineer

FIRST EDITION
FIRST THOUSAND

NEW YORK
JOHN WILEY & SONS
LONDON: CHAPMAN & HALL, LIMITED
1911

THE SCIENTIFIC PRESS
ROBERT DRUMMOND AND COMPANY
BROOKLYN, N. Y.

PREFACE

Previous to the past year very few text-books in English had been published on the subject of statically indeterminate stresses, the books of Martin and Hiroi being the only ones with which the author is familiar. For many years the author has had in mind and has been preparing matter for the publication of a text-book which would give the very few underlying principles of the subject and show their application to a number of the problems to which they apply. While the principle of the work of deformation has been freely used by the author in many of the general demonstrations and in the solution of some of the special problems, the determination of the statically indeterminate stresses and reactions has generally been made directly from the proportionality between load and deformation as the basis.

The deformations which an elastic body undergoes when subjected to load may be determined in several ways if the deformations are so small that the loaded structure remains practically similar to the unstressed structure. For a structure built of an elastic material having a unit elongation, for the unit stress used, of 0.0006, the methods of this book would have a high degree of precision, but for a material just as perfectly elastic having a unit elongation of 0.06 they would have no great value.

Several methods of finding the distortions of elastic structures under loading are developed and the importance of the subject, particularly in bridge erection, is shown. The method of finding deflections by means of the work due to an auxiliary load of unity is given much prominence both for solid and open webbed structures, as it appears to be the simplest of all the methods of computing deflections, when the steps in the operation of computing deflections are considered, as they properly should be, a part of computations which must be made in actually building a structure.

The method of writing the equations for unknown stresses, forces, and reactions in terms of certain elastic deformations due to unit loads has led to new formulas for finding these quantities for certain structures. Such formulas

for unknown crown forces for both the solid webbed and braced arch have been published for the first time by the author.

The space devoted to a comparison of the exact and approximate methods of integration in finding the deflections of certain structures it is believed is warranted and should be of value to persons reading this subject for the first time. The author wishes to acknowledge that Art. 68 has been prepared by HENRY W. HODGE, Art. 69 has been prepared from material furnished by PAUL L. WOLFEL, part of Art. 70 has been prepared by the FORT PITT BRIDGE WORKS, and Art. 71 has been prepared by FRANCIS P. WITMER.

In addition to the books previously alluded to the author desires to acknowledge his indebtedness to the works of CAIN, CHURCH, M. A. HOWE, MERRIMAN, MORLEY, and SWAIN.

Much of the work of computation in connection with the numerical solution of the problems of this book has been done by my assistants, H. P. KIRKHAM, WM. F. CARSON, and E. J. SQUIRE.

Throughout the body of the book where methods have been used which are known by the author to be the result of the labors of others, credit for such work has there been given.

The scope of such a text-book as this may be indefinitely extended by many useful examples, but it is believed that enough examples have been given to show how the solution of problems in deflections and statically indeterminate structures may be made in terms of the elastic properties of the materials.

CLARENCE W. HUDSON.

BROOKLYN, N. Y., June 1, 1911.

CONTENTS

CHAPTER I

ELEMENTARY INDETERMINATE FORMS

CHAPTER II

DEFLECTIONS AND REACTIONS OF STRAIGHT STRUCTURES WITH SOLID WEBS

CHAPTER III

DEFLECTIONS AND STRESSES FOR CURVED STRUCTURES WITH SOLID WEBS

CHAPTER IV

ARCHES WITH SOLID WEBBED RIBS

CHAPTER V

DEFLECTIONS OF STRUCTURES WITH EITHER SOLID OR OPEN WEBS

CHAPTER VI

DEFLECTIONS AND STRESSES IN STRUCTURES WITH OPEN WEBS

CHAPTER VII

MOVABLE BRIDGES

CHAPTER VIII

THE ARCH WITH AN OPEN FRAMEWORK WEB

CHAPTER IX

DISTORTIONS OF STRUCTURES AS AFFECTING THEIR ERECTION

CHAPTER X

ADJUSTING DEVICES AND THE NECESSARY ADJUSTMENTS REQUIRED TO MAKE THE FINAL CONNECTIONS FOR IMPORTANT STRUCTURES

CHAPTER XI

MISCELLANEOUS STRUCTURES AND PROBLEMS

DEFLECTIONS AND STATICALLY INDETERMINATE STRESSES

CHAPTER I

ELEMENTARY INDETERMINATE FORMS

ART. 1. INTRODUCTORY

THE conditions of static equilibrium for forces in one plane are generally stated in works on mechanics to be three, viz.:

The algebraic sum of the horizontal forces on a body must be zero, or algebraically $\Sigma H = 0$.

The algebraic sum of the vertical forces on a body must equal zero, or algebraically $\Sigma V = 0$.

The algebraic sum of the moments of all the forces on a body about any point in the plane of the forces must equal zero, or algebraically $\Sigma M = 0$.

The first two conditions are in reality only one, and when so stated become: the algebraic sum of all the forces on a body must be zero, or algebraically $\Sigma F = 0$.

For most of the problems of applied mechanics the solution is simplified by employing the conception of three conditions of equilibrium.

The algebraic statement of these conditions does not require demonstration, as they are the result of all experience in connection with bodies in equilibrium.

A force is said to be completely defined when its direction, point of application, and magnitude are known, that is, there are in general three unknowns involved in the determination of a force.

The loads to which engineering structures may be subjected are known, or may in general be assumed with a reasonable degree of certainty. The other outer forces are the reactions exerted by the supports, and in order that they may be determined from the conditions of static equilibrium *the total number of unknowns for all the reactions cannot exceed three; at the same time, for stability, the supports must be arranged to prevent both horizontal and vertical motion and rotation.*

a, *b* and *c* of Fig. 1 show the same structure *nopq*, subject to the same loading.

In *a*, the structure is assumed to be supported at *n* on a point, or frictionless pin, and therefore at this point the reaction may have any direction, but must pass through the point *n*. This arrangement gives two unknowns, the direction and the amount, for the reaction at this point.

Throughout this book a support indicated by this symbol △ will be assumed to indicate that furnished by a frictionless pin fixed in position.

The support at *o* of *a* is assumed to be that furnished by a point, or frictionless pin, which may freely move parallel to the supporting surface *rs*, but which is held at the same distance from the surface. This arrangement gives one unknown, the amount, for the reaction.

Throughout this book a support indicated by either this ○ or this symbol ⌂ will be considered to be that furnished by a frictionless pin which may move freely in a direction parallel to the supporting surface at the point, and which is held always at the same distance from the supporting surface.

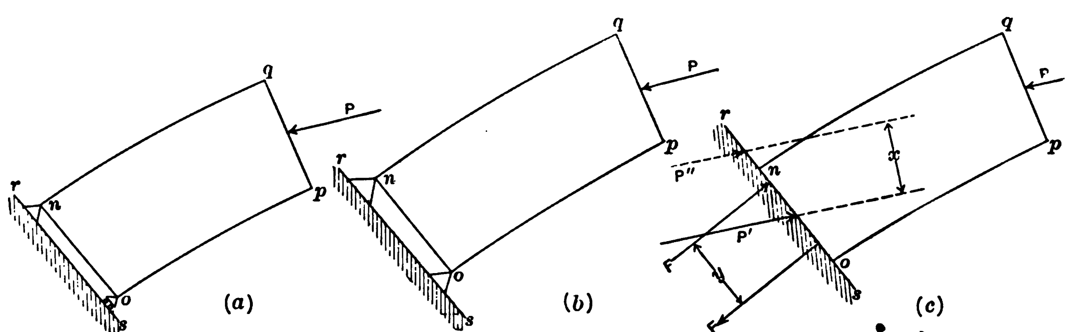


FIG. 1.

For *a* of Fig. 1 the supports are of such a nature as to render the reactions statically determinate, as the total number of unknowns for the reactions is three, and these can be determined from the three equations given from the three conditions of static equilibrium. If the supports at *n* and *o* are interchanged, the structure is also statically determinate. If, however, the support at *n* is made the same as at *o*, the structure is not stable for a general condition of loading, for if the plane *rs* be considered horizontal, the supports are not capable of furnishing the requisite horizontal force to make $\Sigma H = 0$.

If the structure *nopq* be supported as indicated in *b* of Fig. 1, there are two unknowns for each reaction, or four in all, and the reactions are said to be statically indeterminate. If the structure be an elastic one, an additional condition may be written from the known fact that the reactions must be such as to make the total change in length of the structure between the points *n* and *o* equal to zero.

Let the structure *nopq* be regarded as a solid and supported throughout its length along the line *no*, as shown in *c* of Fig. 1, then the exact determination of the reactions, that is, of the stress distribution along the plane *no* of the body furnished by the plane *rs* of the support is very difficult. The accuracy with which this distribution may be determined depends entirely on our ability to express algebraically the law between stress and deformation. For a supporting surface perfectly rigid and the material of the body perfectly elastic, the stress distribution on the surface *no* could be very accurately determined from the theory of flexure. Without going into the exact nature of the stress distribution on this plane *no*, it may be said that if a force P', which is equal and opposite to the resultant P of all the loads on the body, be applied as indicated and that if a couple Fy equal and opposite to the couple Px be applied, that equilibrium will be maintained. It can also be stated that the resultant of the force P' and the couple Fy is the force indicated by the dotted arrow P'', which is equal and opposite to P and has the same line of action as P.

As has been illustrated by *b* and *c* of Fig. 1, unless the total number of the unknowns of the reactions be limited to three they cannot be found from the laws of static equilibrium.

For concurrent coplanar forces the conditions of static equilibrium are reduced to two, viz.: $\Sigma H = 0$ and $\Sigma V = 0$. *Therefore for any jointed structure in order that the stresses in the bars may be determined from the laws of statics, it is necessary that the number of unknown bar stresses and reactions does not exceed twice the number of joints; and in order that the structure may be stable under all conditions of loading the elementary truss figure must be a triangle.*

It may now be stated that a structure may be statically indeterminate, either with reference to the outer forces or inner stresses, or with reference to both outer forces and inner stresses.

Wherever the elastic properties of the material or materials of which a statically indeterminate structure suitable for engineering purposes is built, are known, equations of condition, imposed by either the nature of the supports or the form of the structure itself, together with the three equations of static equilibrium, may be written, from which all the reactions and stresses may be found.

Any reaction condition or structural member in excess of the number required to satisfy the requirements of static equilibrium will be defined as redundant or indeterminate for the purpose of this book.

PROBLEM

No. 1*a*. For the structure indicated in *c* of Fig. 1. Assume $P = 10{,}000$ lbs., $x = 60$ ins., and $y = 30$ ins.; compute F and show graphically by combining F, F and P' that P'' is equal and opposite to P and has the same line of action as P.

ART. 2. ELEMENTARY CONCEPTIONS

A statically indeterminate structure will be further defined as one in which the elastic properties of the materials must be used in order to determine the reactions or the amount of stress in the various parts or members. There are a number of methods, or at least different names for what are essentially the same method, for finding the stresses in structures for which the three equations of static equilibrium do not furnish sufficient equations of conditions to enable the stresses to be found. When a solid body is subjected to external forces, the dimensions of the body are found to change. It is the object of the Mechanics of Materials to study the effect of forces in producing change of shape in bodies

TABLE No. 2*a*

MODULUS OF ELASTICITY FOR VARIOUS MATERIALS

Material.	Tension.	Compression.	Shear.
Wrought iron: Range	24 to 29 millions	24 to 29 millions	
Average	25,000,000	25,000,000	10,000,000
Structural steel	27 to 32 millions	27 to 32 millions	11 to 13 millions
	30,000,000	30,000,000	12,000,000
Nickel steel	27 to 33 millions	27 to 33 millions	
	30,000,000	30,000,000	12,000,000
Cast iron	13 to 22 millions	13 to 25 millions	5.5 to 9 millions
	15,000,000	15,000,000	6,000,000
White oak			
	1,500,000	1,500,000	400,000
Yellow pine			
	1,500,000	1,500,000	400,000
Timber			
(Rough average)	1,000,000	1,000,000	400,000
Granite	....	5 to 13 millions	
	....	8,000,000	
Sandstone	....	1 to 6 millions	
	....	4,000,000	
Limestone	....	1.5 to 8 millions	
	....	5,000,000	
Stone			
(Rough average)	....	6,000,000	
Concrete 1:2:4	....	1 to 3 millions	
(Rough av.) for low unit stresses	....	2,500,000	
Concrete 1:3:6	....	0.8 to 3 millions	
(Rough av.) for low unit stresses	....	2,000,000	
Brick	....	1 to 3.5 millions	
(Rough average)	....	2,000,000	
Aluminium	7 to 11 millions	7 to 11 millions	
	8,000,000	8,000,000	3,500,000
Glass			
	9,000,000		
Phosphor-bronze			
	13,000,000	13,000,000	5,000,000
Copper	15 to 17 millions	15 to 17 millions	
	16,000,000	16,000,000	

and where possible to formulate the experience thus gained into laws. For many of the materials of engineering, strain is directly proportional to the stress which produces it. And it is to structures built of such material that the methods of this book apply.

A bar of medium steel 1 sq. in. in cross section and 10 ft. long, when subjected to an axial tensile stress of 10,000 lbs., elongates .043 inch. A stress twice as great produces twice the elongation, etc.

The ratio of unit stress to unit strain is designated as the modulus of elasticity in works on Mechanics. Table No. 2*a* gives the approximate value of this modulus for the different materials for the three cases of simple stress.

PROBLEM

No. 2*a*. What is the stress in bar of the above material 2 in. square and 5 ft. long when forced between two fixed bearings 59.957 ins. apart?

ART. 3. COMPOSITE MEMBERS

As the modulus of elasticity is the stress which, under the usual assumptions, would elongate a bar of 1 sq.in. cross-section of the given material to double its length, it is seen that this modulus is a measure of the stiffness of the material.

A column composed of four materials is placed in a testing machine, and subjected to a pressure of 100,000 lbs., between two rigid plane surfaces. (See Fig. 3*a*.) The lengths of each of the cylinders will always be the same from the assumed nature of the end planes. Knowing the modulus of elasticity for the different materials, the stresses in P_1 to P_4 may be obtained.

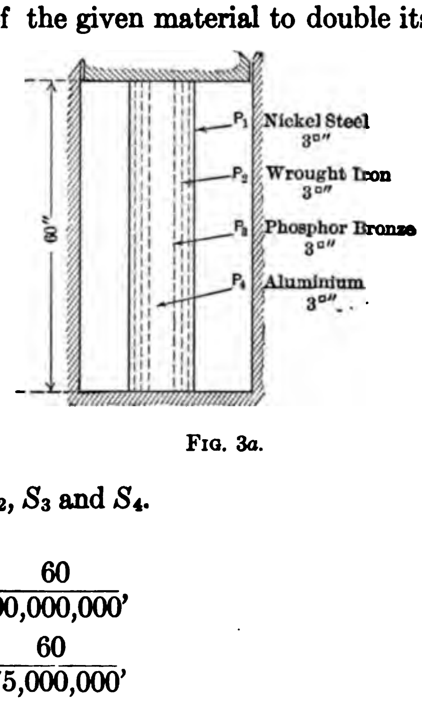


FIG. 3*a*.

Let the stresses in P_1, P_2, P_3 and P_4 be S_1, S_2, S_3 and S_4.

A stress of unity would shorten

$$P_1 \text{ an amount} = \frac{1\times 60}{3\times 30{,}000{,}000} = \frac{60}{90{,}000{,}000},$$

$$P_2 \quad \text{"} \quad = \frac{1\times 60}{3\times 25{,}000{,}000} = \frac{60}{75{,}000{,}000},$$

$$P_3 \quad \text{"} \quad = \frac{1\times 60}{3\times 13{,}000{,}000} = \frac{60}{39{,}000{,}000},$$

$$\text{and } P_4 \quad \text{"} \quad = \frac{1\times 60}{3\times 8{,}000.000} = \frac{60}{24{,}000{,}000}.$$

The shortening of the different materials under their actual stresses are

$$\text{For } P_1=\frac{60S_1}{90,000,000}=\frac{1,560S_1}{2,340,000,000},$$

$$P_2=\frac{60S_2}{75,000,000}=\frac{1,872S_2}{2,340,000,000},$$

$$P_3=\frac{60S_3}{39,000,000}=\frac{3,600S_3}{2,340,000,000},$$

$$P_4=\frac{60S_4}{24,000,000}=\frac{5,850S_4}{2,340,000,000}.$$

These from the nature of the problem are equal, and also the sum of the stresses equals the load,

$$S_1+S_2+S_3+S_4=100,000,$$

or

$$S_1+\frac{1560S_1}{1872}+\frac{1560S_1}{3600}+\frac{1560S_1}{5850}=100,000,$$

or

$$S_1+.833S_1+.433S_1+.267S_1=100,000,$$

$$S_1=\frac{100,000}{2.533}=39,500,$$

and solving similarly

$$S_2=\frac{100,000}{3.040}=32,900,$$

$$S_3=\frac{100,000}{5.846}=17,100,$$

$$S_4=\frac{100,000}{9.500}=10,500.$$

A member in a bridge which during the passage of the live load will be subject to alternate tensile and compressive stress is designed for the tensile stress of 2,000,000 at a unit stress of 20,000 lbs. net section. If the member as designed consists of a riveted part of 60 sq.ins. gross, giving 50 sq.ins. net section and several eye-bars of 50 sq.ins. gross and net, what is the actual tension on the net section of the riveted member if one-sixth of the length of the riveted member is taken out for rivet holes?

The pins at the end may be taken as rigid.

Let S_1 be the total stress carried by the built part of the member, and S_2 be the total stress carried by the eye-bars.

A stress of unity will elongate the built part an amount

$$=\frac{l}{E\times6\times50}+\frac{5l}{E\times6\times60}=\frac{31l}{1800E},$$

and will elongate the eye-bar part an amount

$$=\frac{l}{50E}.$$

The actual changes in length of the built and eye-bar parts of the member are $\frac{31S_1l}{1800E}$ and $\frac{S_2l}{50E}$ respectively, and these from the nature of the problem must be equal. Also

$$S_1+S_2=2,000,000.$$

Substituting for S_2 its value $\frac{31}{36}S_1$, we have:

$$S_1+\frac{31}{36}S_1=2,000,000,$$

and

$$S_1=1,074,600.$$

The tension in the net section $=\frac{1,074,600}{50}=21,500$ per sq.in.

The maximum unit stress on this section is probably considerably greater than 21,500 lbs., as it is believed that built up sections of a given net area do not have as great strength as a bar of uniform section of the given area, and that built tension members do not have the same strength per unit of net area as eye-bars.

PROBLEMS

No. 3*a*. A column of concentric annular portions of concrete of 15 sq.ins., aluminum 10 sq.ins., phosphor bronze 6 sq.ins., wrought iron 3 sq.ins., and high carbon steel 3 sq.ins., is stressed by a total load of 200,000 lbs. What part of the load is carried by each material?

No. 3*b*. A bridge member consists of six eye-bars 8 ins. $\times$1 in. $=$48 sq.ins. gross and net, and a built-up portion of 36 sq.ins. gross, giving 28 sq.ins. net section. The gross section is 7/9 and the net section 2/9 of the length for the built-up part. Under a stress in the member of 1,500,000 lbs. what is the unit stress at the net section of the built-up part?

ART. 4. JOINTED INDETERMINATE FORMS

Let the members T_1, T_2, T_3, and T_4 of Fig. 4*a*, be assumed to be in a vertical plane and to be connected to points, in a horizontal plane, about which they may freely rotate, and that they connect to a similar point and support a load P at their bottom ends.

This problem is very similar to those of Articles 2 and 3, though at first glance it may not seem so.

Let T_1, T_2, T_3, and T_4 be plain rods of structural steel. It is perfectly clear that no matter what the actual stresses are, owing to the symmetry of the figure, the stress in T_1 =stress in T_4 and in $T_2=T_3$.

Experience in connection with tests of steel bars and observation of steel structures under passage of loads teaches beyond all question that the deformation of the structure of Fig. 4a will be very slight, provided P be such as to produce stresses less than the elastic limit of the material.

The elongation of a bar 30 ft. long when stressed up to the elastic limit will not exceed $\frac{1}{4}$ in. The drop of the load P for any rationally designed structure similar to Fig. 4a will not exceed the maximum stretch of a bar 30 ft. long.

Let the members of Fig. 4a be so divided as to form two structures as shown in b and c of Fig. 4, each of which is statically determined. Let the portion of P carried by b be denoted by P_1 and the portion carried by c be denoted by P_2.

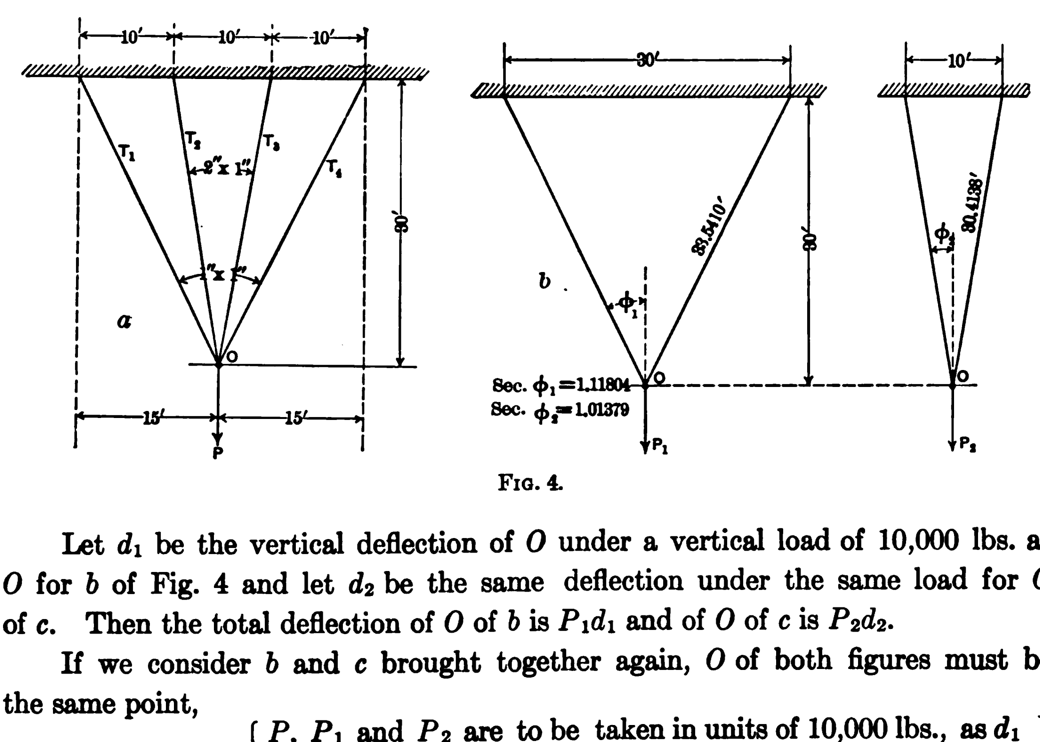


FIG. 4.

Let d_1 be the vertical deflection of O under a vertical load of 10,000 lbs. at O for b of Fig. 4 and let d_2 be the same deflection under the same load for O of c. Then the total deflection of O of b is P_1d_1 and of O of c is P_2d_2.

If we consider b and c brought together again, O of both figures must be the same point,

$\therefore\ P_1d_1 = P_2d_2$
and as $P_1 + P_2 = P$

P, P_1 and P_2 are to be taken in units of 10,000 lbs., as d_1 and d_2 are computed for 10,000 lbs., in order that these deflections may be stated in a small number of decimal places.

All the data to solve the problem when the numerical values of d_1 and d_2 are known is given. These values will be found. The stress in T_1 and T_4 under a vertical load of 10,000 at O of b of Fig. 4 is 5590, and their elongation

$$= \frac{5590 \times 33.5410}{30{,}000{,}000 \times 1} = 0.00625 \text{ ft.}$$

The lengths of the two sides are now $33.5410 + .00625 = 33.54725$ each. As the sides are equal the new position of O is vertically below its former position as is shown in Fig. 4*d*, and the desired deflection

$$d_1 = H - 30,$$

and

$$H = \sqrt{(33.54725)^2 - (15)^2} = 30.00696,$$

$$\therefore \; d_1 = 0.0070 \text{ ft.}$$

The deflection d_2 of O of c is found in the same manner to be .0026 ft., whence

$$P_1 = .271P, \qquad P_2 = .729P,$$

$$T_1 \text{ and } T_4 = +.154P \quad \text{and} \quad T_2 \text{ and } T_3 = +.368P.$$

For $P = 50{,}000$

$$T_1 \text{ and } T_4 = +7700 \quad \text{and} \quad T_2 \text{ and } T_3 = +18400.$$

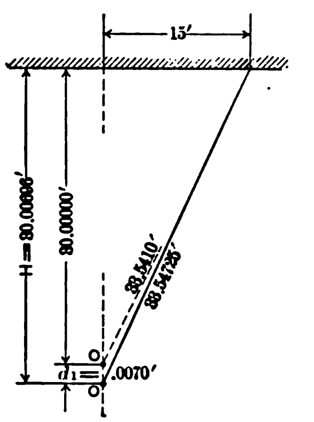


FIG. 4*d*.

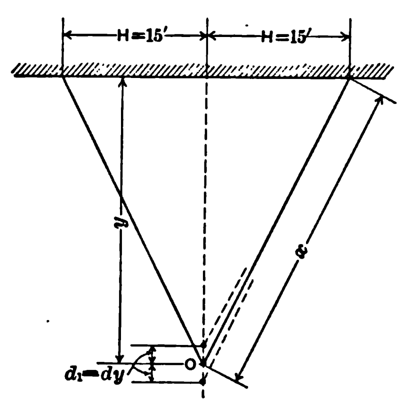


FIG. 4*e*.

The deflections d_1 and d_2 may always be computed, as has just been done, but the method is far too laborious.

The increment or decrement to y due to a corresponding increment or decrement to x may be readily obtained by the use of the calculus.

The general relation between x and y of Fig. 4*e* is $y^2 = x^2 - H^2 = x^2 - 225$,

$$dy = \frac{x}{y} dx.$$

That is, to obtain the deflection, multiply the change in length of x by the sec. ϕ_1.

$dy = d_1 = .00625 \times 1.118 = .0070$, as was obtained by the method of geometry. When the deflected figure does not differ greatly from the original figure the ratio of $\frac{x}{y}$ is almost exactly the same in both cases.

PROBLEM

No. 4*a*. If a vertical member T_5, consisting of a 2 in. $\times$ 2 in. square bar, be added to Fig. 4*a*, and the load $P = 100{,}000$ lbs., compute the stresses in T_1, T_2, T_3, T_4, and T_5.

ART. 5. JOINTED INDETERMINATE FORMS

The determination of the stresses of the structure of Fig. 4*a* was very much simplified by making $T_1 = T_4$ and $T_2 = T_3$ in cross-section. Let T_1, T_2, T_3, and T_4 be $1 \times \frac{1}{2}$ in., 1×1 in., $1\frac{1}{2} \times 1$ in., and 2×1 in. in cross-section, respectively. Let it be required to find the stresses in the various members. The portions of P carried by each of the two statically determined figures, *a* and *b* of Fig. 5, into which the main figure is divided, is now very different from that of Fig. 4*a*. It is clear that any vertical load P acting on *a* will elongate T_1 four times as much as it will T_4 and that therefore the point O will take some new position such as is shown by the intersection of the dotted lines.

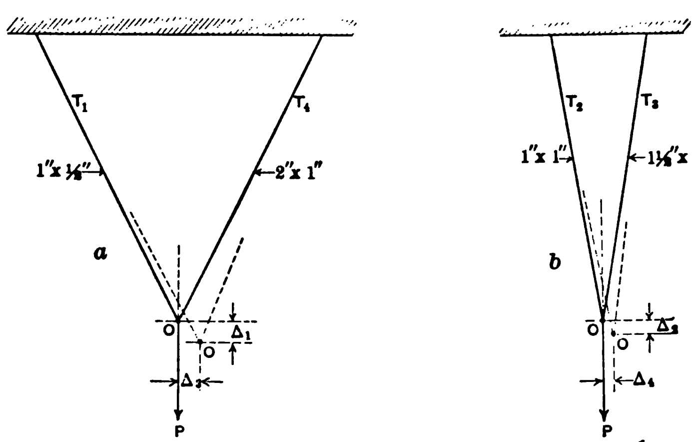


FIG. 5.

In the same manner P will cause the point O of *b* to take some position shown in the same manner. It can readily be shown by geometrical computation for different loadings that for loads producing stresses less than the elastic limit the horizontal and vertical motions of the points O are proportional to the loads that produce them.

Let P_1 (see Fig. 5) be the portion of P carried by *a* when *a* and *b* act together;

P_2 (see Fig. 5) be the portion of P carried by *b* when *a* and *b* act together.;

Δ_1 be the vertical deflection of O produced by P when P is carried entirely by *a*;

Δ_3 be the horizontal deflection of O produced by P when P is carried entirely by *a*;

d_1 be the vertical deflection of O of *a* due to a vertical load of unity at O;

d_3 be the horizontal deflection of O of a due to a vertical load of unity at O;

d_{10} be the vertical deflection of O of a due to a horizontal load of unity at O;

d_{30} be the horizontal deflection of O of a due to a horizontal load of unity at O;

d_2 be the vertical deflection of O of b due to a vertical load of unity at O;

d_4 be the horizontal deflection of O of b due to a vertical load of unity at O;

d_{20} be the vertical deflection of O of b due to a horizontal load of unity at O;

d_{40} be the horizontal deflection of O of b due to a horizontal load of unity at O;

now $d_3 = d_{10}$ and $d_4 = d_{20}$, which does not as yet appear.

Since the ratio of the horizontal to the vertical motion of O of a due to a vertical load is not in general equal to the same ratio for O of b, there will be a horizontal component of the force produced by connecting O of a to O of b, when under vertical loading.

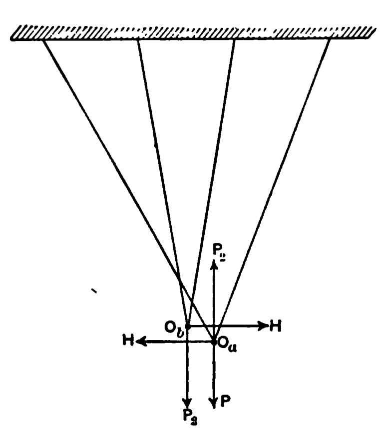


FIG. 5c.

It is clear that the final stresses in a when O of a and b are connected are those due to P when a acts alone modified by P_2 and H, the forces generated at O of a by joining O of a to O of b, and the stresses in b are those due to P_2 and H.

That is, to join Oa to Ob of Fig. 5c, when Oa is supporting P, will develop a force the horizontal component of which may be represented by H and the vertical component by P_2. This problem may be solved in many ways and simpler equations than the following may be written, but it is desirable to make as general a solution as possible.

Three equations will now be written from which P_1, P_2, and H will be found.

Deflections of point O when a is supposed to carry P.

$$\text{The downward deflection of } Oa = \Delta_1 - P_2 d_1 - H d_{10},$$

$$\text{The downward deflection of } Ob = P_2 d_2 + H d_{20}.$$

These must be equal when the structure acts as a whole.

$$\therefore \quad P_2(d_1+d_2)+H(d_{10}+d_{30})=\Delta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The deflection to the right of $Oa=\Delta_3-P_2d_3-Hd_{30}$,

The deflection to the right of $Ob=P_2d_4+Hd_{40}$,

$$\therefore \quad P_2(d_3+d_4)+H(d_{30}+d_{40})=\Delta_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$P_1+P_2=P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From these three equations all the unknown quantities may be found in terms of easily determined deflections.

$$H=\frac{\Delta_1(d_3+d_4)-\Delta_3(d_1+d_2)}{(d_{10}+d_{20})(d_3+d_4)-(d_{30}+d_{40})(d_1+d_2)} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$P_2=\frac{\Delta_1(d_{30}+d_{40})-\Delta_3(d_{10}+d_{20})}{(d_1+d_2)(d_{30}+d_{40})-(d_3+d_4)(d_{10}+d_{20})} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$P_1=P-P_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These may be written in simpler form by making

$$d_1+d_2=a,$$
$$d_3+d_4=b,$$
$$d_{10}+d_{20}=a_1,$$
$$d_{30}+d_{40}=b_1,$$

Then

$$H=\frac{\Delta_1b-\Delta_3a}{a_1b-ab_1},$$

$$P_2=\frac{\Delta_1b_1-\Delta_3a_1}{ab_1-a_1b}.$$

In order to save labor it will be well to tabulate the results of the various steps of the computations as they are made.

Table No. 5a

COMPUTATIONS OF STRESSES IN THE MEMBERS OF FIG. 4a WHEN THE CROSS-SECTION OF THE MEMBERS ARE AS SHOWN IN FIG. 5

Members.	Stresses Due to U_v.	Stresses Due to U_h.	$\lambda_v \times 1000$.	$\lambda_h \times 1000$.	Stresses Due to $P=50000$.	Stresses Due to H.	Stresses Due to $P_1=33333\frac{1}{3}$.	Final Stresses.
T_1	+0.559	+1.118	+0.001293	+0.002586	+27,950	0	−18,630	+ 9,320
T_2	+0.507	+3.041	+0.000532	+0.003190	......	0	+16,900	+16,900
T_3	+0.507	−3.041	+0.000354	−0.002124	......	0	+16,900	+16,900
T_4	+0.559	−1.118	+0.000323	−0.000646	+27,950	0	−18,630	+ 9,320

U_v = vertical load of unity;

U_h = horizontal load of unity acting to the right;

λ_v = elongations for stresses due to U_v;

λ_h = elongations for stresses due to U_h.

E in computing λ_v and λ_h is taken at 29,000,000.

The first six columns of Table No. 5*a* may be computed at once. Having thus determined the changes in length of the various members due to horizontal and vertical forces of unity at O, the deflections d_1 to d_4 and d_{10} to d_{40} will be determined by geometry.

For a sufficient degree of accuracy, to make the determination of the amount of the deflections valuable by this method, the length of the members should be taken to at least five figures for the decimal part, the computation of d_{20} only will be recorded for illustration.

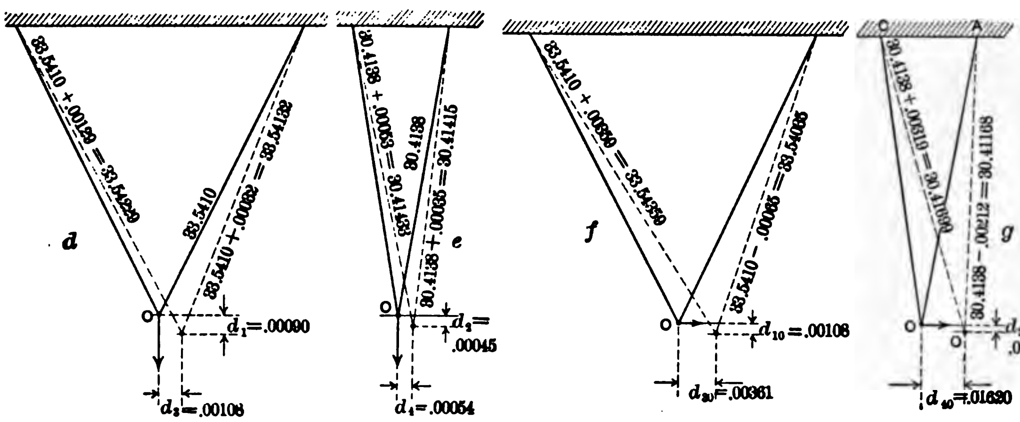


FIG. 5.

The sides and angles used in computing d_{20} are those of the dotted figure of g of Fig. 5:

$$d_{20} = \sin A \times AO - 30.00000 \text{ ft.},$$

$$\sin A = \frac{2\sqrt{S(S-AC)(S-CO)(S-AO)}}{AC \times AO},$$

in which

$$S = \frac{AC+AO+CO}{2} = 35.41435.$$

$$\therefore \quad \operatorname{Sin} A \,.\, AO = \frac{2\sqrt{S(S-AC)(S-CO)(S-AO)}}{AC \times AO} \,.\, AO = \frac{2\sqrt{S(S-AC)(S-CO)(S-AO)}}{AC}$$

$$\log S = 1.549179,$$

$$\log S - AC = 1.405079,$$

$$\log S-CO = .698739,$$

$$\log S-AO = \frac{.699201}{4.352198},$$

$$\text{This} \div 2 = 2.176099.$$

$$\log\left(\frac{2}{AC}=\frac{1}{5}\right) = \underline{.698970}$$

$$\log = 1.477129$$

$$\text{number} = 30.00055 \text{ ft.} = \sin A \, . \, AO,$$

whence

$$d_{20} = .00055 \text{ ft.}$$

a more exact calculation gives $d_{20} = .00054$ ft.

With the deflections found by geometry and of the amounts stated on d, e, f, and g of Fig. 5, the unknowns P_2 and H may be found:

$$\Delta_1 = .00090 \times \frac{50{,}000}{1000} = .045 \text{ ft.},$$

$$\Delta_3 = .00108 \times \frac{50{,}000}{1000} = .054 \text{ ft.},$$

$$(d_1+d_2) = (.00090+00045) \times \frac{1}{1000} = .00000135 \text{ ft.} = a,$$

$$(d_3+d_4) \qquad = .00000162 \qquad = b,$$

$$(d_{10}+d_{20}) \qquad = .00000162 \qquad = a_1,$$

$$(d_{30}+d_{40}) \qquad = .00001981 \qquad = b_1.$$

Whence

$$H = \frac{.045 \times .00000162 - .054 \times .00000135}{.00000162 \times .00000162 - .00001981 \times .00000135} = \frac{729-729}{.026244-.267435} = 0$$

and

$$P_2 = \frac{.045 \times .00001981 - .054 \times .00000162}{.00000135 \times .00001981 - .00000162 \times .00000162} = \frac{8914.5-874.8}{.267435-.026244}$$

$$= \frac{8039.7}{.241191} = 33{,}333\tfrac{1}{3},$$

and

$$P_1 = 16666.7.$$

The correctness of these values for H, P_1, and P_2 may be seen by noting that

$$d_1 = 2d_2 \text{ from } (d) \text{ and } (e) \text{ of Fig. 5},$$

$$d_3 = 2d_4 \text{ from } (d) \text{ and } (e) \text{ of Fig. 5}.$$

The true stresses are now given in the final column of Table No. 5*a*. The fact that $H = 0$ is a somewhat remarkable coincidence, as the data for the problem were taken at random.

Let us suppose T_2 and T_3 of Fig. 5 to be interchanged in position; the complete tabulated computation of the stresses is as follows:

TABLE No. 5b

Members.	Stresses Due to U_v.	Stresses Due to U_h.	$\lambda_v \times 1000$.	$\lambda_h \times 1000$.	Stress Due to 50000.	Stresses Due to $H=1840$.	Stress Due to $P_2=32600$.	Final Stress.
T_1	+0.559	+1.118	+0.001293	+0.002586	+27,950	−2050	−18,200	+ 7,700
T_3	+0.507	+3.041	+0.000532	+0.002124	......	+5580	+16,500	+22,080
T_2	+0.507	−3.041	+0.000354	−0.003190	......	−5580	+16,500	+10,920
T_4	+0.559	−1.118	+0.000323	−0.000646	+27,950	+2050	−18,200	+11,800

The expressions for H and P_2, using values of d_{20} and d_4 with signs opposite to that heretofore, are:

$$H=\frac{\Delta_1(d_3-d_4)-\Delta_3(d_1+d_2)}{(d_{10}-d_{20})(d_3-d_4)-(d_{30}+d_{40})(d_1+d_2)}=\frac{-486}{-.264519}=+1840,$$

and

$$P_2=\frac{\Delta_1(d_{30}+d_{40})-\Delta_3(d_{10}-d_{20})}{(d_1+d_2)(d_{30}+d_{40})-(d_3-d_4)(d_{10}-d_{20})}=\frac{8622.9}{.264519}=+32600$$

PROBLEMS

No. **5a.** Find the stresses in the members of Fig. 5*h* by dividing the main figure into the two static figures *i* and *j* by deriving new general expressions for P_2 and H, when T_1, T_2, T_3, and T_4 are 1 in. × ½ in., 1 in. × 1 in., 1 in. × 1½ in., and 1 in. × 2 in. respectively in cross-section.

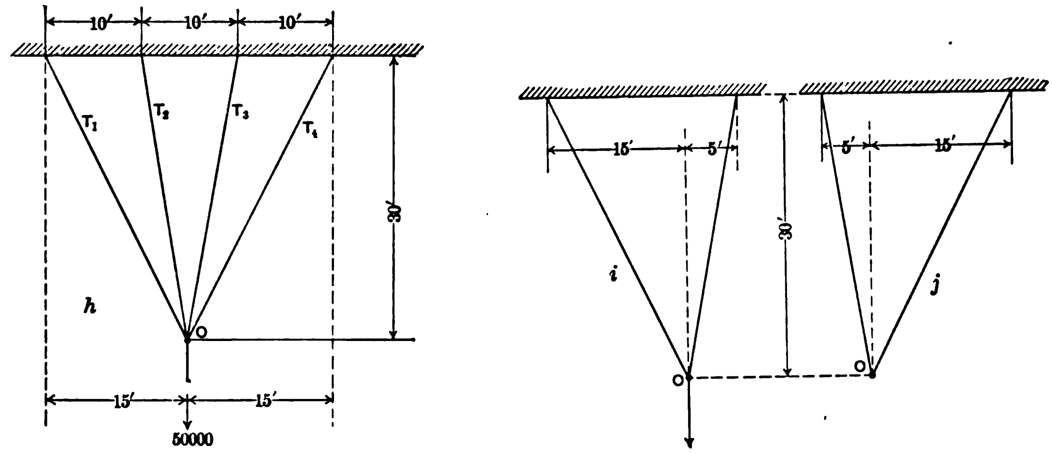


FIG. 5.

No. **5b.** The same as No. 5a except that T_1, T_2, T_3, and T_4 are 1 in. × ½ in., 1½ in. × 1 in., 1 in. × 1 in. and 1 in. × 2 in. respectively.

No. **5c.** What are the final horizontal and vertical motions of point O of Problem No. 5*a*?

No. **5d.** What are the final horizontal and vertical motions of point O of Problem No. 5*b*?

No. **5e.** Compute by geometry Δ_1 and Δ_3 of Fig. 5*a* when $P=25{,}000$ lbs.

No. **5f.** Compute by geometry Δ_1 and Δ_3 of Fig. 5*a* when $P=50{,}000$ lbs.

ART. 6. BEAMS AND TRUSSES

In a structure which rests on three supports of the nature indicated in Fig. 6*a*, the conditions of static equilibrium do not furnish enough conditions to enable the reactions R_1, R_2, and R_3 to be determined.

If the interior support be considered slowly lowered until it no longer touches the structure, the structure deflects until the internal stresses in its component parts are sufficient to establish equilibrium and there is given a structure on two supports, or a structure statically determinate with reference to the exterior forces.

Now with the center support considered removed, if we can find the deflection of b, with reference to its former position under the action of any loads, P, which will be called Δ and if at the same time we can find the deflection of b under a load of unity at b, which will be called d, then $R_2 = \frac{\Delta}{d}$ for an elastic structure.

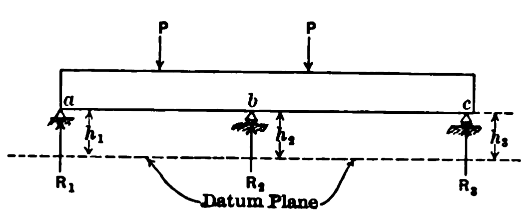


FIG. 6a.

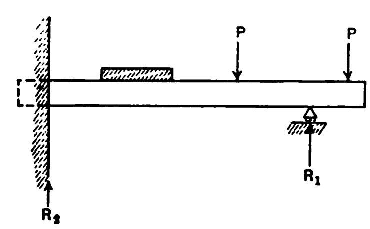


FIG. 6b.

In a structure fixed at one end and supported at the other such as is shown in Fig. 6*b*, to determine the reaction R_1 and thereby R_2, consider the support at R_1 removed and let Δ be the deflection under the given loading of this point of support, then if d be the deflection of the point of support under a load of unity when R_1 is removed, $R_1 = \frac{\Delta}{d}$ for the given loading and the support in place.

As has been previously stated, any support to a structure other than those required by the condition of static equilibrium, renders the outer forces indeterminate. This indeterminateness may properly be considered as due to the inability of the structure to take the natural distortions it would undergo if subject to statically determinate reactions. Suppose the structure of Fig. 6*r* to be held in equilibrium by two supports capable of exerting only vertical reactions. The distorted structure assumes somewhat the form of the dotted line figure of *s*. If the structure is supported in the manner indicated, in either *t*, *u*, or *v*, the distorted figure will be somewhat like that of the respective dotted line figures of each case.

Each support fixed in position is capable of exerting a reaction which must pass through the point of support, and may always be represented by horizontal and vertical component forces. These component forces for each reaction are a direct function of the horizontal and vertical displacements which would take place under the given loading if the support in question were removed, provided the structure is perfectly elastic and the displacements very small.

Further, as the displacements at any point in a structure are in general a function of all the loads, the displacements at each support, when enough redundant conditions have been considered removed to render the structure determinate, may be written in terms of the loads and the component forces at the other supports. By writing equations for the displacements at each point of support in succession as stated, a sufficient number of equations, taken with the three static conditions, may be written to enable all the unknown forces to be found. For structures which contain more members than are necessary for stability, the stresses in the redundant bars may be found the same as unknown outer forces, subject to the limits of the problem.

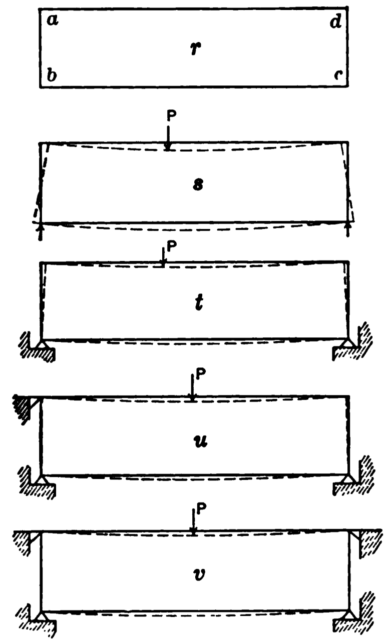


FIG. 6.

The method of thus determining the indeterminate forces and stresses will be illustrated by many examples in a later part of this book.

The exactness of the solution of statically indeterminate structures depends on the strictness of the relation between loads and their deformations and nothing else.

It may be stated that the design of a structure statically indeterminate either with reference to the outer or inner forces may be made, provided expressions for the redundant forces may be written in terms of the elastic properties of the materials.

As the solution of problems in connection with statically indeterminate structures depends on the determination of the elastic deformations of the structure, taken under such conditions as are imposed by the nature of the problem, a study of the best means of determining elastic deformations is absolutely essential. For this and the further reason that a knowledge of the elastic deflections of structures is essential for the proper handling of much engineering construction, considerable space will be devoted to this subject in the following pages. For-

tunately many works on Applied Mechanics and Structures contain enough on the subject to make additional study easy.

PROBLEM

No. 6*a*. Consult any text-book on Applied Mechanics or a later part of this book, and from the equation of the axis of a beam determine the center deflection Δ of the beam due to a load P at a distance kl from the left end (k being a fraction of l). Then find the center deflection d due to a load of unity at the center. From the values of Δ and d thus found, derive a general formula for the center reaction for a continuous beam on level supports with two equal spans and loaded at any point.

CHAPTER II

DEFLECTIONS AND REACTIONS OF STRAIGHT STRUCTURES WITH SOLID WEBS

ART. 7. DEFLECTIONS OF A STRAIGHT BEAM BY MEANS OF THE DIFFERENTIAL EQUATION OF ITS AXIS

THE curve formed by the axis of a bar when subject to flexure is called the elastic line. There are several general differential equations of this axis. The equation of the elastic line, which perhaps has the most general applicability to the problems of flexure, is $EI.\frac{d^2y}{dx^2}=M$, in which M is the moment of the stress couple on any cross-section expressed as a function of x, the effect of the shearing stresses in producing distortion being neglected.

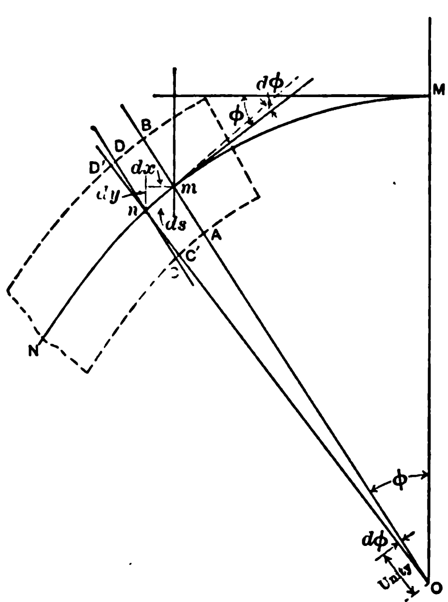


FIG. 7a.

Let the dotted portion of Fig. 7a represent a part of any beam subject to flexure of which the full line MN is the neutral axis, and let AB and CD be any two plane sections distant ds before bending. Let the relative motion of these two planes, which takes place under flexure, be considered with reference to the section AB. Then

$$ds=r.d\phi,$$

and from experience $ds=dx$ with a high degree of accuracy, for all the materials used in flexure in engineering practice. Whence

$$r=\frac{dx}{d\phi},$$

and further $d\phi=d\left(\frac{dy}{dx}\right)$, for such values of ϕ as will be met in engineering practice.

$$\therefore \quad r=\frac{dx}{d\left(\frac{dy}{dx}\right)}=\frac{dx}{\frac{d^2y}{dx}}=\frac{dx^2}{d^2y}.$$

From Fig. 7*a*, the angles *mon* and DnD' being equal, we have

$$om : mn :: nD : DD',$$

or

$$r : dx :: c : e,$$

in which c is the distance from the neutral axis of the piece to the fiber whose elongation is e. e for this case, where S is the unit stress of the fiber,

$$= \frac{SL}{E} = S\frac{dx}{E},$$

whence

$$r = \frac{cdx}{e} = cdx\frac{E}{Sdx} = \frac{cE}{S}, \quad \text{and} \quad \frac{c}{S} = \frac{I}{M}.$$

$$\therefore \quad r = \frac{EI}{M}, \quad \text{and} \quad \frac{dx^2}{d^2y} = \frac{EI}{M}, \quad \text{or} \quad EI\frac{d^2y}{dx^2} = M.$$

For any special beam, M in the preceding differential equation is stated in terms of x measured from a selected origin and the expression integrated twice, the constants of integration being determined for the limits of the special case.

ART. 8. DEFLECTIONS OF STRAIGHT BEAMS BY MEANS OF THE WORK DUE TO AN AUXILIARY LOAD OF UNITY [1]

The differential equation of the elastic line given in Art. 7, while the most general in its applicability to the problem of flexure for straight beams, is on account of its form not the one best suited to certain of these problems. In many cases only the deflection at a certain point of a structure is desired without the necessity of knowing the deflection at all points. The general expression for the deflection at any definite point of a straight beam of length l subject to flexure will now be derived.

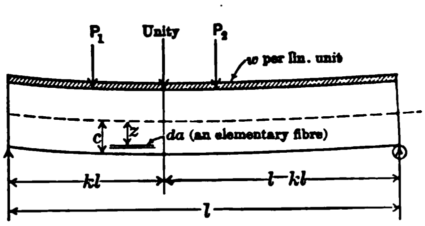


FIG. 8*a*.

Let Δ = the deflection of any point in the desired direction for a given loading.

Let kl = distance from some known point to the point whose deflection is required, k being a fraction and less than unity.

Let unity be an auxiliary force acting at the point and in the direction for which Δ is desired.

Let M be the bending moment at any point on the neutral axis produced by the given loading.

[1] The formula of this article was first derived by Professor Fraenkel.

Let m be the bending moment due to the auxiliary load of unity at the point where M is taken.

Then the stress in any fiber of an area da and distant z from the neutral axis due to M

$$=\frac{Mc}{I}\cdot\frac{z}{c}\cdot da,$$

and the same due to m

$$=\frac{mc}{I}\cdot\frac{z}{c}\cdot da.$$

The elongation of any fiber for a length dx due to M is

$$\frac{Mc}{I}\cdot\frac{z}{c}\cdot da\cdot\frac{dx}{Eda}=\frac{M}{I}\cdot z\cdot\frac{dx}{E}.$$

The work in this fiber of length dx due to moving the fiber stress for the auxiliary load of unity when gradually applied through the elongation of the fiber due to M

$$=\frac{M}{I}\cdot z\cdot\frac{dx}{E}\cdot\frac{m}{2I}\cdot z\cdot da=\frac{Mmz^2da}{2EI^2}\cdot dx,$$

and for the entire cross-section of the beam the work

$$=\frac{Mmdx}{2EI},$$

and the internal work throughout the beam

$$=\int_0^l\frac{Mmdx}{2EI}.$$

The external work of the auxiliary force in moving through the deflection due to the given loading

$$=\tfrac{1}{2}\times\Delta=\frac{\Delta}{2}.$$

The external and internal work are equal.

$$\therefore\quad \Delta=\int_0^l\frac{Mmdx}{EI}=\int_0^{kl}\frac{Mmdx}{EI}+\int_{kl}^l\frac{Mmdx}{EI}.$$

In the derivation of the previous general formula if M be taken as the bending moment due to a load of unity acting in the same manner as the load of unity which develops the moment m then the value of $\Delta=\int_0^l\frac{m^2dx}{EI}$.

It may seem clearer to the student to include the deformation due to the auxiliary load of unity in writing the expressions for both the internal and external work due to this auxiliary force. To thus include the effect of this load,

Let Δ_1 = the deflection of the point of application of the force of unity due to its own action;

λ_1 = the change in length of any fiber due to the force of unity; $=\frac{mzdx}{EI}$, as was shown for the given loading.

The total change of length of any fiber now

$$=\frac{Mzdx}{EI}+\frac{mzdx}{EI}=(M+m)\frac{zdx}{EI},$$

and the internal work throughout the beam,

$$W=\int_0^l \frac{Mmdx}{2EI}+\int_0^l \frac{m^2dx}{2EI}. \qquad (a)$$

The total deflection desired, now $=\Delta+\Delta_1$, and the external work done by the force of unity

$$=(\Delta+\Delta_1)\times\tfrac{1}{2}. \qquad (b)$$

Making Eq. (a) = Eq. (b), we have

$$\Delta+\Delta_1=\int_0^l \frac{Mmdx}{EI}+\int_0^l \frac{m^2dx}{EI};$$

but

$$\Delta_1=\int_0^l \frac{m^2dx}{EI}, \quad \text{therefore} \quad \Delta=\int_0^l \frac{Mmdx}{EI}.$$

PROBLEM

No. 8a. Derive the general expression for the deflection in the direction of the line of action of the load, of the loaded point in a beam under one load.

ART. 9. DEFLECTION OF STRAIGHT BEAMS DUE TO SHEAR

The value of Δ just determined is that due to the longitudinal flexural stresses; this deflection will be increased by the effect of the shearing stresses. The deflection Δ_s due to shear for a rectangular prismatic section will now be determined. Let a and b of Fig. 9 represent a portion of the length, and the cross-section of such a beam. Let S be the shear at any section due to the given loading. Let s be the shear at the same section for the auxiliary unit load.

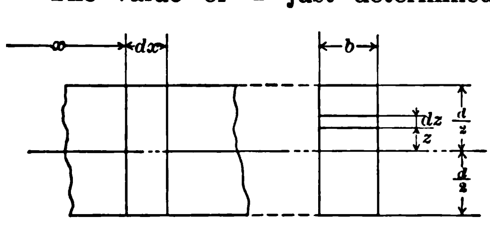


FIG. 9.

Let Q be the intensity of shear at a distance z from the neutral axis due to S.

Let q be the intensity of shear at a distance z from the neutral axis due to s.

Let Q_f be the amount of shear on an elementary area dist. z from the neutral axis due to S.

Let q_f be the amount of shear on an elementary area dist. z from the neutral axis due to s. Then

$$Q = \frac{S}{Ib}\left(\frac{d}{2} - z\right)b\frac{\left(\frac{d}{2} + z\right)}{2} = \frac{S}{2I}\left(\frac{d^2}{4} - z^2\right),$$

and

$$Q_f = \frac{S}{2 \cdot I}\left(\frac{d^2}{4} - z^2\right)bdz.$$

The transverse deformation for an element of length dx

$$\frac{Q_f dx}{E \cdot b \cdot dz} = dx \cdot b \cdot dz \cdot \frac{S}{2I}\frac{\left(\frac{d^2}{4} - z^2\right)}{Ebdz} = dx . \frac{S}{2IE}\left(\frac{d^2}{4} - z^2\right),$$

in which it should be noted that E is the modulus of elasticity for shear.

Also $q_f = \frac{s}{2I}\left(\frac{d^2}{4} - z^2\right) \times b \cdot dz$ and if q_f is gradually developed, the work done by unity on an element of the cross-section and dx in length $= dx . \frac{Ss}{8EI^2}\left(\frac{d^2}{4} - z^2\right)^2 b \cdot dz.$ For the section chosen, this is readily integrable. For an I section, the cross-section should be divided up into several parts and the average values for S and s for each part used in determining the work on the entire section and dx in length.

For the rectangular beam taken, the work on the entire cross-section for a length dx

$$= dx \int_{-\frac{d}{2}}^{+\frac{d}{2}} \frac{Ss}{8EI^2} \cdot \left(\frac{d^2}{4} - z^2\right)^2 \cdot b \cdot dz = dx . \frac{18Ss}{Ebd^6} \times \left[\frac{d^4 z}{16} - \frac{d^2 z^3}{6} + \frac{z^5}{5}\right]_{-\frac{d}{2}}^{+\frac{d}{2}} = dx . \frac{18Ss}{30Ebd}.$$

Now let δ_{dx} be the vertical deflection of the unit load caused by the shearing stress on the elementary length, then if unity be gradually applied the external work $= \frac{\delta_{dx}}{2}$ and this $= dx . \frac{18Ss}{30Ebd}$, or

$$\delta_{dx} = dx . \frac{18}{15}\frac{Ss}{Ebd} = dx . \frac{6}{5}\frac{Ss}{EA},$$

and for the whole length

$$\Delta_s = \int_0^l dx . \frac{6}{5} \cdot \frac{Ss}{EA}.$$

In the immediately following is given the maximum deflection due to the shearing stresses for four cases of loading for a beam of rectangular cross-section.

Case I. Cantilever with concentrated load at end.

$$\Delta_s = \frac{6}{5}\int_0^l \frac{Ssdx}{E_s A} = \frac{6}{5}\int_0^l \frac{P1dx}{E_s A}$$

$$= \frac{6}{5}\left[\frac{Px}{E_s A}\right]_0^l = \frac{6Pl}{5E_s A}.$$

Case II. Cantilever with uniform load of w per unit of length.

$$\Delta_s=\frac{6}{5}\int_0^l\frac{w\cdot x\cdot l\cdot dx}{E_sA}=\frac{6}{5}w\Big|_0^l\left[\frac{x^2}{2E_sA}\right]=\frac{6wl^2}{10E_sA}.$$

Case III. Simple beam with concentrated load at the center.

$$\Delta_s=\frac{6}{5}\int_0^{\frac{l}{2}}\frac{\frac{P}{2}\cdot\frac{1}{2}\cdot dx}{E_sA}+\frac{6}{5}\int_{\frac{l}{2}}^{l}\frac{\left(-\frac{P}{2}\right)\left(-\frac{1}{2}\right)dx}{E_sA}=\frac{6P}{20}\Big|_0^l\left[\frac{x}{E_sA}\right]=\frac{6Pl}{20E_sA}.$$

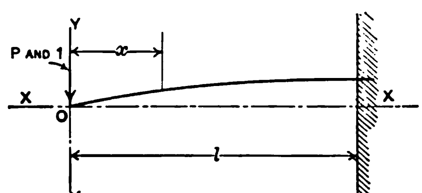


Fig. 9 c

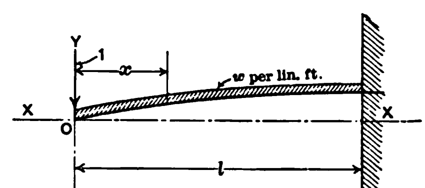


Fig. 9 d

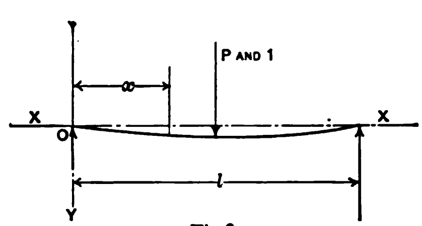


Fig. 9 e

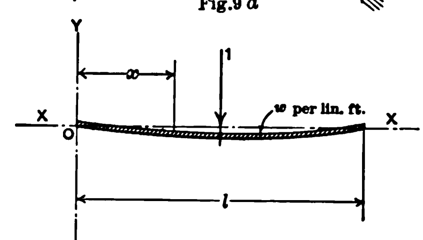


Fig. 9 f

Case IV. Simple beam with uniform load of w per unit of length.

$$\Delta_s=\frac{6}{5}\int_0^{\frac{l}{2}}\frac{\left(\frac{wl}{2}-wx\right)\cdot\frac{1}{2}\cdot dx}{E_sA}+\frac{6}{5}\int_{\frac{l}{2}}^{l}\frac{\left(\frac{wl}{2}-wx\right)\cdot\left(-\frac{1}{2}\right)dx}{E_sA}$$

$$=\frac{6}{5E_sA}\Big|_0^{\frac{l}{2}}\left[\frac{wlx-wx^2}{4}\right]+\frac{6}{5E_sA}\Big|_{\frac{l}{2}}^{l}\left[\frac{-wlx+wx^2}{4}\right]$$

$$=\frac{6}{20E_sA}\left[\frac{wl^2}{2}-\frac{wl^2}{4}\right]+\frac{6}{20E_sA}\left[-wl^2+wl^2+\frac{wl^2}{2}-\frac{wl^2}{4}\right]$$

$$=\frac{6}{20E_sA}\left[\frac{wl^2}{4}+\frac{wl^2}{4}\right]$$

$$=\frac{6}{20E_sA}\cdot\frac{wl^2}{2}=\frac{6wl^2}{40E_sA}.$$

PROBLEM

No. 9*a*. Find the deflection at the center, due to shear, of a 24-in.-80-lb.-I beam 24 ft. long, loaded at the center with 40,000 lbs., and compare this to deflection due to the longitudinal flexural stresses.

ART. 10. DEFLECTION OF A STRAIGHT CANTILEVER BEAM FOR A LOAD AT ANY POINT, THE MOMENT OF INERTIA BEING CONSTANT

The equation of the elastic line of a cantilever beam under a concentrated load at any point will now be determined.

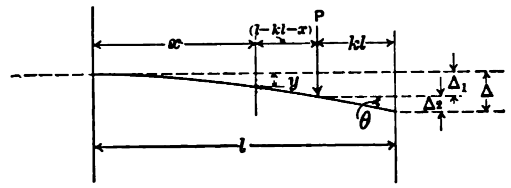


FIG. 10*a*.

Let Fig. 10*a* represent such a beam. The equation will consist of two parts. The elastic line for the portion kl is straight. For the curved portion to the left of P we have

$$EI\frac{d^2y}{dx^2}=M=-P(l-x-kl)=-Pl+Px+Pkl,$$

and integrating

$$EI\frac{dy}{dx}=-Plx+\frac{Px^2}{2}+Pklx+C$$

but for $x=o$, $\frac{dy}{dx}=0$, and therefore $C=0$, and hence

$$EI\frac{dy}{dx}=-Plx+\frac{Px^2}{2}+Pklx.$$

Integrating, again

$$EIy=-P\frac{lx^2}{2}+\frac{Px^3}{6}+Pk\frac{lx^2}{2}+C,$$

but for $x=0$, $y=0$, and therefore $C=0$, and hence

$$y=\frac{Px^2}{6EI}(x-3l+3kl),$$

which is the equation of the line for all points to the left of P, and calling Δ_1 the deflection at the load where

$$x=(l-kl)=l(1-k),$$

then

$$\Delta_1=\frac{Pl^2(1-k)^2}{6EI}[l(1-k)-3l+3kl]=\frac{Pl^2(1-k)^2}{6EI}[-2l+2kl]=-\frac{Pl^3}{3EI}[1-k]^3.$$

The deflection at any point to the right of P and dist. kl from P and l from the wall $=+(-\Delta_1)+(-\Delta_2)=\Delta=-\frac{Pl^3}{3EI}[1-k]^3-kl\tan\theta.$

Now $\tan\theta=\frac{dy}{dx}$ when $x=l-kl$.

It has been seen that

$$EI\frac{dy}{dx} = -Plx + \frac{Px^2}{2} + Pklx,$$

and this gives for $\frac{dy}{dx}$ when $x = l - kl = l(1-k)$

$$\frac{dy}{dx} = \frac{P}{EI}\left[-l^2(1-k) + \frac{l^2(1-k)^2}{2} + kl^2(1-k)\right] = \frac{Pl^2}{EI}(1-k)\left(-1 + \frac{(1-k)}{2} + k\right)$$

$$= \frac{Pl^2}{EI}(1-k)\frac{(-1+k)}{2} = -\frac{Pl^2}{2EI}(1-k)^2.$$

$$\therefore\ \Delta = -\frac{Pl^3}{3EI}[1-k]^3 - \frac{Pl^2}{2EI}(1-k)^2 kl = -\frac{Pl^3}{6EI}(1-k)^2[2(1-k)+3k]$$

$$= -\frac{Pl^3}{6EI}[2-2k+3k](1-k)^2 = -\frac{Pl^3}{6EI}(1-k)^2(2+k)$$

$$= -\frac{Pl^3}{6EI}(1-2k+k^2)(2+k) = -\frac{Pl^3}{6EI}[2-4k+2k^2+k-2k^2+k^3]$$

$$= -\frac{Pl^3}{6EI}(2-3k+k^3).$$

ART. 11. DEFLECTION OF A STRAIGHT SIMPLE BEAM FOR A LOAD AT ANY POINT, THE MOMENT OF INERTIA BEING CONSTANT

Let Fig. 11*a* show such a beam, the notation being sufficiently explained by the figure.

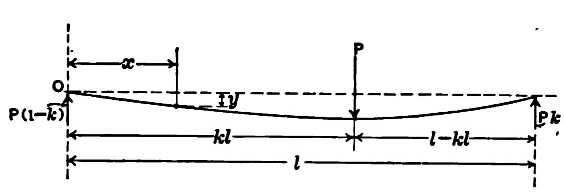


FIG. 11*a*.

On the left of the load

$$EI\frac{d^2y}{dx^2} = P(1-k)x, \qquad (1)$$

$$EI\frac{dy}{dx} = P(1-k)\frac{x^2}{2} + C_1, \qquad (2)$$

$$EIy = P(1-k)\frac{x^3}{6} + C_1x + C_2, \qquad (3)$$

$C_2 = 0$, for $x = 0$ when $y = 0$.

On the right of the load

$$M = P(1-k)x - P(x-kl) = Px - Pkx - Px + Pkl = Pk(l-x),$$

$$EI\frac{d^2y}{dx^2} = Pk(l-x) = Pkl - Pkx, \quad (4)$$

$$EI\frac{dy}{dx} = Pklx - \frac{Pkx^2}{2} + C_3, \quad (5)$$

$$EIy = \frac{Pklx^2}{2} - \frac{Pkx^3}{6} + C_3x + C_4, \quad (6)$$

$$C_4 = -\frac{Pkl^3}{2} + \frac{Pkl^3}{6} - C_3l,$$

from (6) for $y=0$ when $x=l$.

$$C_4 = -\frac{Pkl^3}{3} - C_3l, \quad (7)$$

$$\frac{P(1-k)k^2l^2}{2} + C_1 = Pk^2l^2 - \frac{Pk^3l^2}{2} + C_3, \text{ from (2) and (5)}, \quad (8)$$

$$\frac{P(1-k)k^3l^3}{6} + C_1kl = \frac{Pk^3l^3}{2} - \frac{Pk^4l^3}{6} + C_3kl - \frac{Pkl^3}{3} - C_3l, \text{ from (3),(6) and (7)}. \quad (9)$$

Dividing (9) by kl,

$$\frac{P(1-k)k^2l^2}{6} + C_1 = \frac{Pk^2l^2}{2} - \frac{Pk^3l^2}{6} + C_3 - \frac{Pl^2}{3} - \frac{C_3}{k}. \quad (10)$$

Subtracting (10) from (8),

$$\frac{P(1-k)k^2l^2}{3} = \frac{Pk^2l^2}{2} + \frac{Pl^2}{3} - \frac{Pk^3l^2}{3} + \frac{C_3}{k},$$

and

$$2P(1-k)k^3l^2 = 3Pk^3l^2 + 2Pkl^2 - 2Pk^4l^2 + 6C_3,$$

or

$$6C_3 = 2Pk^3l^2 - 2Pk^4l^2 + 2Pk^4l^2 - 2Pkl^2 - 3Pk^3l^2 = -Pk^3l^2 - 2Pkl^2,$$

$$\therefore \quad C_3 = -\frac{Pk^3l^2}{6} - \frac{Pkl^2}{3}, \quad (11)$$

$$\frac{P(1-k)k^2l^2}{2} + C_1 = Pk^2l^2 - \frac{Pk^3l^2}{2} - \frac{Pk^3l^2}{6} - \frac{Pkl^2}{3},$$

from (8) by substituting the value of C_3 in (11), or

$$C_1 = -\frac{Pk^2l^2}{2} + \frac{Pk^3l^2}{2} + Pk^2l^2 - \frac{Pk^3l^2}{2} - \frac{Pk^3l^2}{6} - \frac{Pkl^2}{3}$$

$$= \frac{Pk^2l^2}{2} - \frac{Pk^3l^2}{6} - \frac{Pkl^2}{3} = -\frac{Pl^2}{6}(2k - 3k^2 + k^3).$$

Substituting C_1 in (3), remembering that $C_2 = 0$, we have

$$EIy = \frac{P}{6}(1-k)x^3 - \frac{P}{6}(2k - 3k^2 + k^3)l^2x$$

$$y = \frac{P}{6EI}(1-k)x^3 - \frac{P}{6EI}(2k - 3k^2 + k^3)l^2x. \quad . \quad . \quad . \quad . \quad (12)$$

Let k be supposed greater than $\frac{1}{2}$, then the maximum deflection will be somewhere to the left of the load. To find where the deflection is a maximum for any special value of k, substitute in (12) the value of k and form the first derivative of y with respect to x and place equal to 0 and solve for x.

From (12)

$$\frac{dy}{dx} = 0 = \frac{3P}{6EI}(1-k)x^2 - \frac{Pl^2}{6EI}(2k - 3k^2 + k^3),$$

and

$$x^2 = \frac{k(1-k)(2-k)l^2}{3(1-k)} = \left(\frac{2k}{3} - \frac{k^2}{3}\right)l^2 \quad \text{and} \quad x = l\left(\frac{2k}{3} - \frac{k^2}{3}\right)^{\frac{1}{2}}.$$

Substituting this value of x in (12) and calling the maximum value of y, y_1, we have:

$$y_1 = -\frac{Pl^3}{3EI}(1-k)\left(\frac{2k}{3} - \frac{k^2}{3}\right)^{\frac{3}{2}}.$$

Substituting in Eq. (6) the values determined for C_3 and C_4 there is given

$$EIy = \frac{Pklx^2}{2} - \frac{Pkx^3}{6} - \frac{Pk^3l^2x}{6} - \frac{Pkl^2x}{3} + \frac{Pk^3l^3}{6},$$

as the equation of the elastic line on the right of the load. When $x = \frac{l}{2}$,

$$y = \frac{Pl^3}{48EI}(4k^3 - 3k). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

When $k = \frac{1}{2}$ and $x = \frac{l}{2}$,

$$y = -\frac{Pl^3}{48EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

ART. 12. THE THEOREM OF THREE MOMENTS

One of the most useful applications of the equation of the elastic line is in finding the reactions for continuous beams. To show one of the most general applications of this let Fig. 12*a* show any two adjacent spans of a continuous beam having a constant moment of inertia.

The beam is considered to be without longitudinal restraint, that is, that the reactions for any system of vertical loading are vertical, and also that the position of the neutral axis at the supports are maintained always at the same elevation.

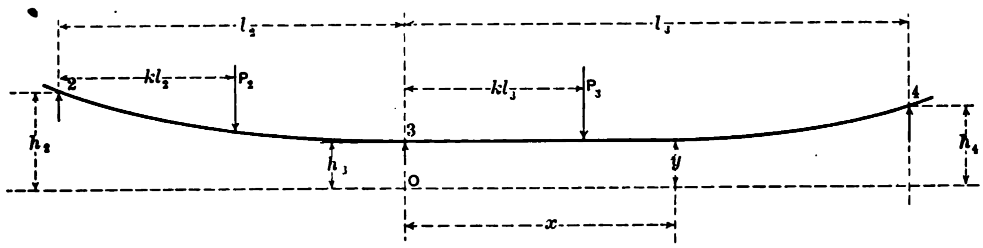


FIG. 12*a*.

Let the heavy line of Fig. 12*a*, represent the neutral axis of any two adjacent spans of a continuous beam.

Let M denote the bending moment at any point, and M_2, M_3 and M_4 be the ding moments at supports 2, 3 and 4 respectively.

V_2, V_3 and V_4 be the shears just to the right of supports 2, 3 and 4 respect-

remainder of the notation is sufficiently defined by the figure.

$$\frac{d^2y}{dx^2}=\frac{M}{EI}. \qquad \text{(1)}$$

$$M=M_3+V_3x-P_3(x-kl_3), \qquad \text{(2)}$$

in general for any point to right of P_3. When $x=l_3$, M becomes M_4.

$$\therefore \quad M_4=M_3+V_3l_3-P_3l_3+P_3kl_3,$$

and

$$V_3=\frac{M_4-M_3}{l_3}+P_3(1-k), \qquad \text{(3)}$$

also

$$M=M_3+V_3x,$$

in general for any point to the left of P_3.

Substituting this valve of M in (1),

$$\frac{d^2y}{dx^2}=\frac{M_3+V_3x}{EI}, \quad \text{or} \quad EI\frac{d^2y}{dx^2}=M_3+V_3x.$$

Integrating, we have

$$EI\frac{dy}{dx}=M_3x+\tfrac{1}{2}V_3x^2+C_1, \quad \ldots \ldots \quad (4)$$

$$EIy=\tfrac{1}{2}M_3x^2+\tfrac{1}{6}V_3x^3+C_1x+C_2. \quad \ldots \ldots \quad (5)$$

Substituting in (1) the value of M in (2) we have

$$EI\frac{d^2y}{dx^2}=M_3+V_3x-P_3(x-kl_3),$$

and integrating,

$$EI\frac{dy}{dx}=M_3x+\tfrac{1}{2}V_3x^2-\tfrac{1}{2}P_3x^2+P_3kl_3x+C_3, \quad \ldots \ldots \quad (4')$$

$$EIy=\tfrac{1}{2}M_3x^2+\tfrac{1}{6}V_3x^3-\tfrac{1}{6}P_3x^3+\tfrac{1}{2}P_3kl_3x^2+C_3x+C_4, \quad \ldots \quad (5')$$

when $x=0$ in (5), $y=h_3$,

$$\therefore \quad C_2=EIh_3.$$

When $x=l_3$ in (5′) $y=h_4$,

$$\therefore \quad C_4=EIh_4-\tfrac{1}{2}M_3l_3^2-\tfrac{1}{6}V_3l_3^3+\tfrac{1}{6}P_3l_3^3-\tfrac{1}{2}P_3kl_3^3-C_3l_3.$$

Substituting in (5) and (5′) the values of C_2 and C_4 respectively, we have,

$$EIy=\tfrac{1}{2}M_3x^2+\tfrac{1}{6}V_3x^3+C_1x+EIh_3, \quad \ldots \ldots \quad (5_1)$$

$$EIy=\tfrac{1}{2}M_3(x^2-l_3^2)+\tfrac{1}{6}V_3(x^3-l_3^3)-\tfrac{1}{6}P_3(x^3-l_3^3)+C_3(x-l_3)$$
$$+EIh_4+\tfrac{1}{2}P_3kl_3(x^2-l_3^2), \quad \ldots \ldots \quad (5_1')$$

when $x=kl_3$, in (4) and (4′), $\frac{dy}{dx}$ is the same in both equations.

$$\therefore \quad M_3kl_3+\tfrac{1}{2}V_3k^2l_3^2+C_1=M_3kl_3+\tfrac{1}{2}V_3k^2l_3^2-\tfrac{1}{2}P_3k^2l_3^2+P_3k^2l_3^2+C_3, \quad \ldots \quad (a)$$

when $x=kl_3$ in (5_1) and $(5_1')$ y is the same in both equations.

$$\therefore \quad \tfrac{1}{2}M_3k^2l_3^2+\tfrac{1}{6}V_3k^3l_3^3+C_1kl_3+EIh_3=\tfrac{1}{2}M_3l_3^2(k^2-1)$$
$$+\tfrac{1}{6}V_3l_3^3(k^3-1)-\tfrac{1}{6}P_3l_3^3(k^3-1)+C_3l_3(k-1)+\tfrac{1}{2}P_3kl_3^3(k^2-1)+EIh_4. \quad \ldots \quad (a_1)$$

From (a)

$$C_1-C_3=P_3k^2l_3^2-\tfrac{1}{2}P_3k^2l_3^2=\frac{P_3k^2l_3^2}{2},$$

and

$$\begin{cases} C_1=C_3+\tfrac{1}{2}P_3k^2l_3^2, \quad \ldots \ldots \quad (b) \\ C_3=C_1-\tfrac{1}{2}P_3k^2l_3^2. \quad \ldots \ldots \quad (b_1) \end{cases}$$

From (a_1),

$$C_1kl_3-C_3kl_3+C_3l_3=-\tfrac{1}{2}M_3l_3^2-\tfrac{1}{6}V_3l_3^3-\tfrac{1}{6}P_3l_3^3(k^3-1)+\tfrac{1}{2}P_3kl_3^3(k^2-1)+EIh_4-EIh_3.$$

Substituting in (a_1) for C_1 its value in (b), we have

$$C_3kl_3+\frac{P_3}{2}k^3l_3{}^3-C_3kl_3+C_3l_3=-\tfrac{1}{2}M_3l_3{}^2-\tfrac{1}{6}V_3l_3{}^3-\tfrac{1}{6}P_3l_3{}^3(k^3-1)$$
$$+\tfrac{1}{2}P_3kl_3{}^3(k^2-1)+EIh_4-EIh_3,$$

or

$$C_3l_3=-\tfrac{1}{2}M_3l_3{}^2-\tfrac{1}{6}V_3l_3{}^3-\tfrac{1}{2}P_3k^3l_3{}^3-\tfrac{1}{6}P_3k^3l_3{}^3+\tfrac{1}{6}P_3l_3{}^3+\tfrac{1}{2}P_3kl_3{}^3(k^2-1)+EI(h_4-h_3),$$

and

$$C_3=-\tfrac{1}{2}M_3l_3-\tfrac{1}{6}V_3l_3{}^2-\tfrac{1}{6}P_3l_3{}^2(4k^3-1)+\tfrac{1}{2}P_3kl_3{}^2(k^2-1)+\frac{EI(h_4-h_3)}{l_3}.$$

Substituting the value of C_3 in (b_1) for C_3 in (a_1), we have

$$C_1kl_3-C_1kl_3+\tfrac{1}{2}P_3k^3l_3{}^3+C_1l_3-\tfrac{1}{2}P_3k^2l_3{}^3=-\tfrac{1}{2}M_3l_3{}^2-\tfrac{1}{6}V_3l_3{}^3-\tfrac{1}{6}P_3l_3{}^3(k^3-1)$$
$$+\tfrac{1}{2}P_3kl_3{}^3(k^2-1)+EIh_4-EIh_3,$$

or

$$C_1l_3=-\tfrac{1}{2}M_3l_3{}^2-\tfrac{1}{6}V_3l_3{}^3-\tfrac{1}{6}P_3l_3{}^3(k^3-1)+\tfrac{1}{2}P_3kl_3{}^3(k^2-1)-\tfrac{1}{2}P_3k^2l_3{}^3(k-1)+EI(h_4-h_3),$$

and

$$C_1=-\tfrac{1}{2}M_3l_3-\tfrac{1}{6}V_3l_3{}^2-\tfrac{1}{6}P_3l_3{}^2(k^3-1)+\tfrac{1}{2}P_3kl_3{}^2(k^2-1)-\tfrac{1}{2}P_3k^2l_3{}^2(k-1)+\frac{EI(h_4-h_3)}{l^3}.$$

Substituting the value of C_1 in (4),

$$EI\frac{dy}{dx}=M_3x+\tfrac{1}{2}V_3x^2-\tfrac{1}{2}M_3l_3-\tfrac{1}{6}V_3l_3{}^2-\tfrac{1}{6}P_3l_3{}^2(k^3-1)+\tfrac{1}{2}P_3kl_3{}^2(k^2-1)$$
$$-\tfrac{1}{2}P_3k^2l_3{}^2(k-1)+EI\frac{(h_4-h_3)}{l_3}.$$

Substituting in the above for V_3 its value in (3),

$$EI\frac{dy}{dx}=M_3x+\frac{x^2}{2}\left[\frac{M_4-M_3}{l_3}+P_3(1-k)\right]-\tfrac{1}{2}M_3l_3-\frac{l_3{}^2}{6}\left[\frac{M_4-M_3}{l_3}+P_3(1-k)\right]$$
$$-\tfrac{1}{6}P_3l_3{}^2(k^3-1)+\tfrac{1}{2}P_3kl_3{}^2(k^2-1)-\tfrac{1}{2}P_3k^2l_3{}^2(k-1)+EI\frac{(h_4-h_3)}{l_3}.$$

The value of $\frac{dy}{dx}$ at support No. 3 is obtained by making $x=0$.

$$EI\frac{dy}{dx}=-\tfrac{1}{3}M_3l_3-\frac{M_4l_3}{6}+\frac{P_3kl_3{}^2}{6}-\frac{P_3l_3{}^2}{6}-\frac{P_3k^3l_3{}^2}{6}+\frac{P_3l_3{}^2}{6}-\frac{P_3kl_3{}^2}{2}-\frac{P_3k^3l_3{}^2}{2}$$
$$+\frac{P_3k^2l_3{}^2}{2}+EI\frac{(h_4-h_3)}{l_3}+\frac{P_3k^3l_3{}^2}{2},$$

whence

$$\frac{dy}{dx}=\frac{h_4-h_3}{l_3}-\frac{2M_3l_3+M_4l_3+P_3l_3{}^2(k^3-3k^2+2k)}{6EI}, \quad . \quad . \quad . \quad . \quad (6)$$

which is the value of the tangent to the elastic curve at support No. 3.

Substituting the value of C_3 in (4′) we have

$$EI\frac{dy}{dx}=M_3x+\tfrac{1}{2}V_3x^2-\tfrac{1}{2}P_3x^2+P_3kl_3x-\tfrac{1}{2}M_3l_3-\tfrac{1}{6}V_3l_3^2-\tfrac{1}{6}P_3l_3^2(4k^3-1)$$
$$+\tfrac{1}{2}P_3kl_3^2(k^2-1)+EI\frac{(h_4-h_3)}{l_3}.$$

and substituting in the above for V_3 its value in (3),

$$EI\frac{dy}{dx}=M_3x+\tfrac{1}{2}x^2\left[\frac{M_4-M_3}{l_3}+P_3(1-k)\right]-\tfrac{1}{2}P_3x^2+P_3kl_3x-\tfrac{1}{2}M_3l_3$$
$$-\tfrac{1}{6}l_3^2\left[\frac{M_4-M_3}{l_3}+P_3(1-k)\right]-\tfrac{1}{6}P_3l_3^2(4k^3-1)+\tfrac{1}{2}P_3kl_3^2(k^2-1)+EI\frac{(h_4-h_3)}{l_3},$$

and when $x=l_3$,

$$EI\frac{dy}{dx}=M_3l_3+\frac{M_4l_3}{2}-\frac{M_3l_3}{2}+\frac{P_3l_3^2}{2}-\frac{P_3kl_3^2}{2}-\frac{P_3l_3^2}{2}+P_3kl_3^2-\tfrac{1}{2}M_3l_3-\frac{M_4l_3}{6}+\frac{M_3l_3}{6}$$
$$-\frac{P_3l_3^2}{6}-\frac{4P_3k^3l_3^2}{6}+\tfrac{1}{6}P_3l_3^2+\frac{P_3kl_3^2}{6}+\tfrac{1}{2}P_3k_3^3l_3^2-\tfrac{1}{2}P_3kl_3^2-EI\frac{(h_4-h_3)}{l_3}$$
$$=\frac{M_3l_3}{6}+\frac{2M_4l_3}{6}+\frac{P_3kl_3^2}{6}-\frac{P_3k^3l_3^2}{6}+EI\frac{(h_4-h_3)}{l_3},$$

and the value of the tangent to the elastic curve at support No. 4 is when the subscript (3) is added to k to indicate which span it applies to.

$$\frac{dy}{dx}=\frac{h_4-h_3}{l_3}+\frac{M_3l_3+2M_4l_3+P_3l_3^2(k_3-k_3^3)}{6EI}. \quad . \quad . \quad . \quad . \quad . \quad (6')$$

Diminishing all the subscripts in equation (6′) by unity, we have the tangent at support No. 3 in terms of load P_2 on span l_2.

$$\frac{dy}{dx}=\frac{h_3-h_2}{l_2}+\frac{M_2l_2+2M_3l_2+P_2l_2^2(k_2-k_2^3)}{6EI}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The values of $\frac{dy}{dx}$ in (6) and (7) are equal when the subscript three (3) is added to k in (6) as they are for the same point.

$$\therefore\quad \frac{h_3-h_2}{l_2}+\frac{M_2l_2+2M_3l_2+P_2l_2^2(k_2-k_2^3)}{6EI}$$
$$=\frac{h_4-h_3}{l_3}-\frac{2M_3l_3+M_4l_3+P_3l_3^2(k_3^3-3k_3^2+2k_3)}{6EI}. \quad . \quad . \quad (8)$$

If all the supports are at the same elevation (8) reduces to

$$M_2l_2+2M_3l_2+P_2l_2^2(k_2-k_2^3)=-2M_3l_3-M_4l_3-P_3l_3^2(k_3^3-3k_3^2+2k_3),$$

or

$$M_2l_2+2M_3(l_2+l_3)+M_4l_3=-P_2l_2^2(k_2-k_2^3)-P_3l_3^2(k_3^3-3k_3^2+2k_3). \quad . \quad (9)$$

Eq. (8) is the most general form of the relation between the moments at the three supports for any two adjacent spans of a straight or nearly straight prismatic beam, all the quantities of which are known but the moments. One such equation can be written for each support of a continuous beam, hence as many equations as there are unknown moments can be written, and the unknown moments found.

With the moments known at the supports, the reactions may readily be found and the beam investigated by the laws of static equilibrium.

ART. 13. REACTIONS FOR TWO EQUAL SPANS BY THE THEOREM OF THREE MOMENTS.

For two equal spans with a load in the first span only, as indicated in Fig 13a, we have from (9) of Art. 12, when $l_1 = l_2 = l$.

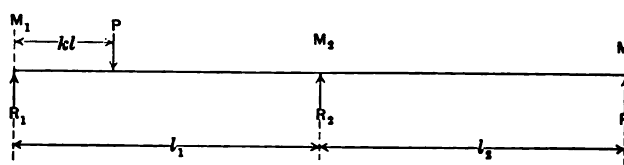


FIG. 13a.

$$4M_2 = -P(k-k^3)l,$$

also

$$M_2 = R_1 l - P(l-kl),$$

from statics

$$\therefore \quad R_1 l - Pl + Pkl + \frac{Pkl}{4} - \frac{Pk^3 l}{4} = 0,$$

$$R_1 \, R_1 = \frac{P}{4}(4+k^3-5k) = P(1-k) - \frac{P}{4}(k-k^3), \quad . \quad . \quad . \quad . \quad . \quad (10)$$

also

$$M_2 = R_3 l.$$

$$\therefore \quad R_3 l + \frac{Pkl}{4} - \frac{Pk^3 l}{4} = 0,$$

$$R_3 = \frac{P}{4}(k^3-k) = -\frac{P}{4}(k-k^3). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now

$$R_1 + R_2 + R_3 = P \qquad \text{or} \qquad R_2 = P - (R_1 + R_3),$$

$$R_2 = P - \frac{P}{4}(4+k^3-5k) - \frac{P}{4}(k^3-k),$$

$$R_2 = \frac{P}{2}(3k-k^3) = Pk + \frac{P}{2}(k-k^3). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

PROBLEM

No. 13a. Compute the amount of R_1, R_2, and R_3 for $P = 10,000$ lbs., $\kappa = \frac{1}{4}$, and $l_1 = l_2 = 10$ ft.

ART. 14. REACTIONS FOR A CONTINUOUS BEAM OF THREE SPANS, BY THE THEOREM OF THREE MOMENTS

For three spans, the end spans being equal and the center span nl with a load in the first span as indicated in Fig. 14*a*, n being the ratio of the centre to the end span.

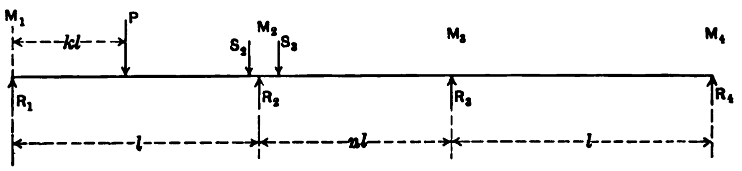


FIG. 14*a*.

$$M_1=0 \quad \text{and} \quad M_4=0.$$

From (9)

$$2M_2(l+nl)+M_3nl=-Pl^2(k-k^3), \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

remembering that

$$M_1=0,$$

and

$$M_2nl+2M_3(nl+l)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

as

$$M_4=0.$$

From (β)

$$M_3=\frac{-M_2nl}{2(nl+l)}.$$

Substituting this in (α), we have

$$2M_2(l+nl)-\frac{M_2n^2l^2}{2(nl+l)}=-Pl^2(k-k^3),$$

$$M_2\left\{2+2n-\frac{n^2l}{2(nl+l)}\right\}=-Pl(k-k^3),$$

$$M_2\left\{\frac{4nl+4l+4n^2l+4nl-n^2l}{2(nl+l)}\right\}=-Pl(k-k^3),$$

$$M_2=-\frac{2(nl+l)\ k-k^3)P}{8n+3n^2+4}=-\frac{2P(nl+l)(k-k^3)}{4(n+1)^2-n^2},$$

$$M_3=\frac{2nlP(nl+l)(k-k^3)}{2(nl+l)\{4(n+1)^2-n^2\}}=\frac{nlP(k-k^3)}{4(n+1)^2-n^2}.$$

Now having the values of M_1, M_2, M_3, and M_4, we can find the values of R_1, R_2, R_3, and R_4.

$$M_2 = R_1 l - P(l - kl) = -\frac{2P(nl+l)(k-k^3)}{4(n+1)^2 - n^2},$$

letting $4(n+1)^2 - n^2 = m$,

$$R_1 = P(1-k) - \frac{2+2n}{m}P(k-k^3), \qquad \ldots \ldots \quad (13)$$

$$M_3 = M_2 + S_3 nl = \frac{nlP(k-k^3)}{m} = -\frac{2P(nl+l)(k-k^3)}{m} + S_3 nl,$$

$$= -\frac{2P(nl+l)(k-k^3)}{m} + (R_2 - P + R_1)nl.$$

$$\therefore \quad \frac{nlP(k-k^3)}{m} = -\frac{2P(nl+l)(k-k^3)}{m} + R_2 nl - Pnl + Pnl(1-k) - \frac{2+2n}{m}Pnl(k-k^3),$$

$$R_2 nl = \frac{2P(nl+l)(k-k^3)}{m} - Pnl(1-k) + Pnl + \frac{2+2n}{m}Pnl(k-k^3) + \frac{nlP(k-k^3)}{m},$$

$$R_2 = \frac{2P(n+1)(k-k^3) - Pnm(1-k) + Pnm + 2(1+n)Pn(k-k^3) + Pn(k-k^3)}{nm}$$

$$= \frac{P(2nk - 2nk^3 + 2k - 2k^3 + knm - nm + nm + 2nk - 2nk^3 + 2n^2k - 2n^2k^3 + nk + nk^3)}{nm}$$

$$= \frac{P(5nk - 5nk^3 + 2k - 2k^3 + knm + 2n^2k - 2n^2k^3)}{nm},$$

$$R_2 = Pk + \frac{2+5n+2n^2}{nm}P(k-k^3) \qquad \ldots \ldots \quad (14)$$

$$M_3 = R_4 l = \frac{nlP(k-k^3)}{m},$$

$$R_4 = \frac{n}{m}P(k-k^3), \qquad \ldots \ldots \quad (15)$$

$$R_3 = P - R_1 - R_2 - R_4 = P - P(1-k) + \frac{2+2n}{m}P(k-k^3) - \frac{n}{m}P(k-k^3)$$
$$- Pk - \frac{2+5n+2n^2}{nm}(k-k^3)P,$$

$$R_3 = \frac{2n + 2n^2 - n^2 - 2 - 5n - 2n^2}{nm}P(k-k^3),$$

$$= -\frac{2+3n+n^2}{nm}P(k-k^3),$$

$$R_3 = -\frac{2+3n+n^2}{nm}P(k-k^3) \qquad \ldots \ldots \quad (16)$$

ART. 15. REACTIONS OF BEAMS ON THREE SUPPORTS BY MEANS OF THE WORK DONE BY AN AUXILIARY LOAD OF UNITY

One of the simplest and most useful of the applications of the method of Art. 8 is that of finding the reactions of the drawbridge of two equal spans. The reactions of such a structure are three and are one more than can be found by the laws of statics. If one of the reactions be found by some other method, then the remaining two may be found from the laws of statics.

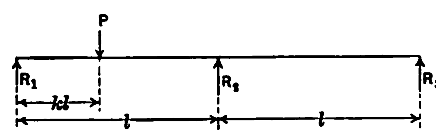


FIG. 15a.

Let Fig. 15a represent a simple beam with the support at the left end considered removed.

Let Δ = the deflection of the left end due to a load P at a distance kl from the left end;

δ = the deflection of the left end due to a load of unity at that end;

R_1 = the reaction at the left end due to P when the support is in place.

Then

$$R_1=\frac{\Delta}{\delta}=\frac{\int_0^{2l} Mmdx\div EI}{\int_0^{2l} mmdx\div EI}=\frac{\int_0^{2l} Mmdx}{\int_0^{2l} m^2dx}$$

$$=\frac{\int_0^{kl} 0.x.dx+\int_{kl}^{l} P(x-kl)xdx+\int_0^{l} P(1-k)x.x.dx}{2\int_0^{l} x.xdx}$$

$$=\frac{P\int_{kl}^{l}\left(\frac{x^3}{3}-\frac{klx^2}{2}\right)+P\int_0^{l}\left(\frac{x^3}{3}-\frac{kx^3}{3}\right)}{2\int_0^{l}\frac{x^3}{3}}$$

$$=\frac{\left[\frac{l^3}{3}-\frac{kl^3}{2}-\frac{k^3l^3}{3}+\frac{k^3l^3}{2}+\frac{l^3}{3}-\frac{kl^3}{3}\right]}{\frac{2.l^3}{3}}$$

$$=\frac{P}{4}[4-5k+k^3],$$

which is the same value as obtained in Art. 13 by special application of the theorem of three moments.

ART. 16. DEFLECTION OF A CANTILEVER BEAM BY MEANS OF THE DIFFERENTIAL EQUATION OF ITS ELASTIC LINE, WHEN THE MOMENT OF INERTIA OF THE BEAM IS NOT CONSTANT

The application of the differential equation of the elastic line to finding the deflection of a beam subject to flexure, when the beam has a varying cross-section will now be shown.

For this purpose let us take a center-bearing plate-girder draw-bridge of two equal arms. This very light, single track, deck structure is shown in Fig. 16a, in sufficient detail for the purpose of illustration. Let it be required to find the deflection at the center of the end bearings.

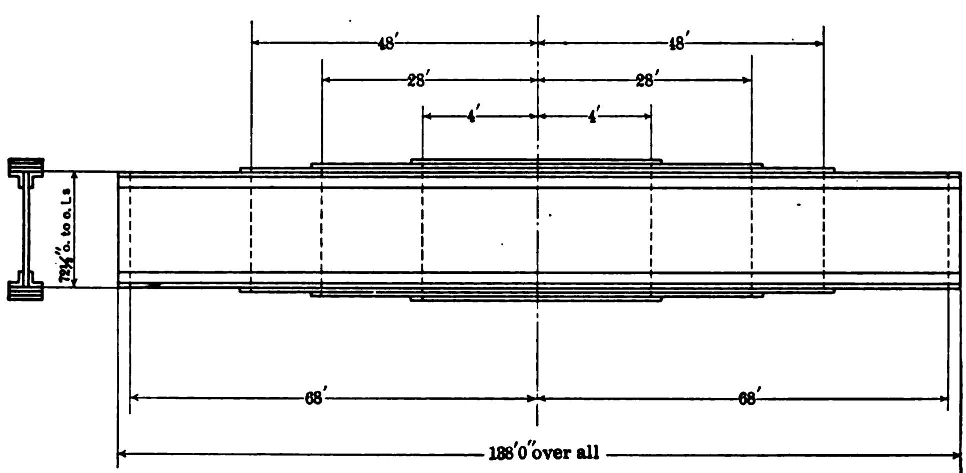


FIG. 16a.

Make up of Girder.		Overall Length.	Effective Length.
1 web pl.	$72 \times \frac{3}{8}$	138 ft.	136 ft.
2 top angles	$5 \times 5 \times 18.33$	138 "	136 "
1 top pl.	$12 \times \frac{5}{8}$	98 "	96 "
1 "	$12 \times \frac{5}{8}$	22 "	20 "
1 "	$12 \times \frac{5}{8}$	10 "	8 "
2 bottom angles	$5 \times 5 \times 18.33$	138 "	136 "
1 bottom plate	$12 \times \frac{5}{8}$	98 "	96 "
1 "	$12 \times \frac{5}{8}$	22 "	20 "
1 "	$12 \times \frac{5}{8}$	10 "	8 "

I_1	38,284 ins.^{4}	1.85 ft.^{4}
I_2	58,335 "	2.81 "
I_3	79,081 "	3.81 "
I_4	100,526 "	4.85 "

As the structure is symmetrical about the center line, the deflection of both arms will be the same, and the tangent to the elastic line at the center will be horizontal. We will assume that the girder has a uniform load of $w = 500$ lbs. per ft. of length. Let Fig. 16b show one-half the deflected girder with notation so shown that further definition is unnecessary.

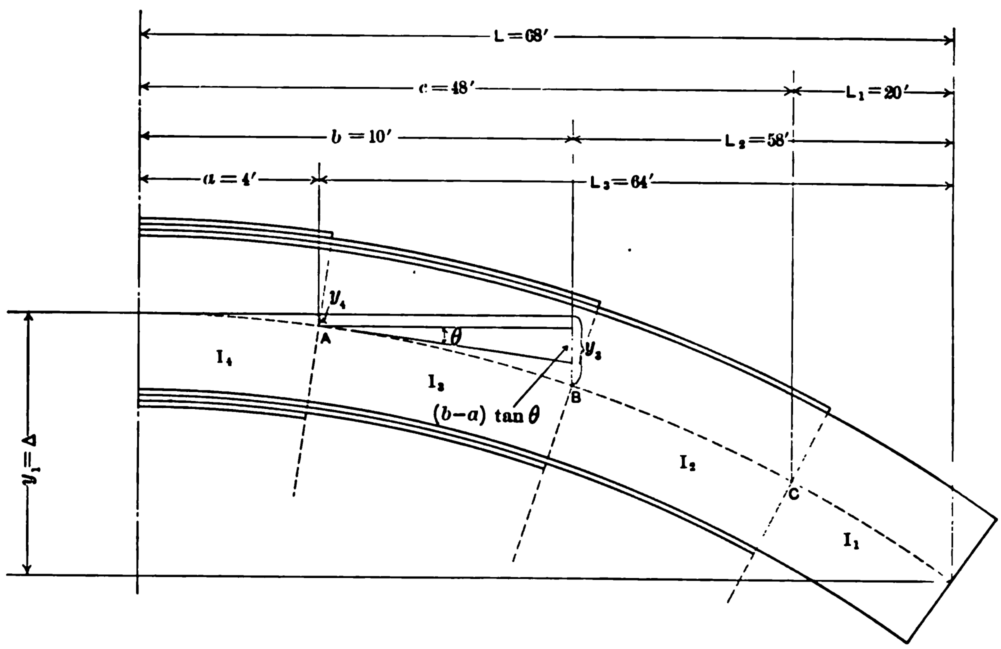


FIG. 16b.

The general equation of the elastic curve is

$$\frac{d^2y}{dx^2} = \frac{M}{EI}.$$

Whence, if L = length of arm and the origin be taken at the center,

$$\frac{d^2y}{dx^2} = \frac{w(L-x)^2}{2EI} = \frac{w(L^2-2Lx+x^2)}{2EI},$$

$$\frac{dy}{dx} = \frac{w}{2EI}\left[L^2x - Lx^2 + \frac{x^3}{3}\right] + C_1,$$

when $x = 0$,

$$\frac{dy}{dx} = 0,$$

$$\therefore\ C_1 = 0,$$

and

$$y = \frac{w}{2EI}\left[\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right] + C_2,$$

when $x = 0$,

$$y = 0,$$

$$\therefore\ C_2 = 0.$$

By aid of the values for $\frac{dy}{dx}$ and y, we will now proceed to find the deflection at the end of the plate-girder draw span. The deflection y_4 at A, the end of the first cover plate, will be the same as if the girder was of constant moment of inertia, I_4, throughout its entire length L. The deflection of a point on the axis at B, the end of the second cover plate, will equal that due to its position on a beam of length $(L-a)$ with uniform load w and constant moment of inertia I_3, plus the deflection caused by the angle θ which the beam axis makes with the horizontal at A, and which is equal to $\frac{dy_4}{dx_4}(b-a)$, plus the distance y_4 which the beam has deflected at A. Continuing this operation at the end of each cover plate or at each change in the moment of inertia we may finally obtain the deflection at the end of the girder. The coordinates of a point on the axis for the portion having a moment of inertia I_4 are y_4 and x_4, and similarly for the coordinates of other points.

$$y_4=\frac{w}{24E}\left[\frac{6L^2x_4^2-4Lx_4^3+x_4^4}{I_4}\right],$$

$$y_3=y_4+\frac{w}{24E}\left[\frac{6L_3^2x_3^2-4L_3x_3^3+x_3^4}{I_3}\right]+\left[\frac{dy_4}{dx_4}\right]x_3,$$

$$y_2=y_3+\frac{w}{24E}\left[\frac{6L_2^2x_2^2-4L_2x_2^3+x_2^4}{I_2}\right]+\left[\frac{dy_3}{dx_3}\right]x_2,$$

$$y_1=y_2+\frac{w}{24E}\left[\frac{6L_1^2x_1^2-4L_1x_1^3+x_1^4}{I_1}\right]+\left[\frac{dy_2}{dx_2}\right]x_1,$$

$$\frac{dy_4}{dx_4}=\frac{w}{2E}\left[\frac{L^2x_4-Lx_4^2+\frac{x_4^3}{3}}{I_4}\right],$$

$$\frac{dy_3}{dx_3}=\frac{w}{2E}\left[\frac{L_3^2x_3-L_3x_3^2+\frac{x_3^3}{3}}{I_3}\right]+\frac{dy_4}{dx_4},$$

$$\frac{dy_2}{dx_2}=\frac{w}{2E}\left[\frac{L_2^2x_2-L_2x_2^2+\frac{x_2^3}{3}}{I_2}\right]+\frac{dy_3}{dx_3}.$$

Substituting in the above equations the values of L, x, and I, we have,

$$\frac{dy_4}{dx_4}=\frac{w}{2E}\left[\frac{\overline{68}^2\times 4-68\times 4^2+\frac{\overline{4}^3}{3}}{4.85}\right]=\frac{3593.7w}{2E},$$

$$\frac{dy_3}{dx_3}=\frac{w}{2E}\left[\frac{\overline{64}^2\times 6-64\times \overline{6}^2+\frac{6^3}{3}}{3.81}\right]+\frac{3593.7w}{2E}=\frac{9458.3w}{2E},$$

$$\frac{dy_2}{dx_2}=\frac{w}{2E}\left[\frac{\overline{58}^2\times 38-58\times \overline{38}^2+\frac{38^3}{3}}{2.81}\right]+\frac{9458.3w}{2E}=\frac{31{,}654.3w}{2E},$$

$$y_4=\frac{w}{24E}\left[\frac{6\times \overline{68}^2\times \overline{4}^2-4\times 68\times \overline{4}^3+4^4}{4.85}\right]=\frac{87{,}990.1w}{24E},$$

$$y_3=\frac{87990.1w}{24E}+\frac{w}{24E}\left[\frac{6\times 64^2\times 6^2-4\times 64\times 6^3+6^4}{3.81}\right]$$

$$+\frac{3593.7w\times 12\times 6}{24E}=\frac{564{,}777.4w}{24E},$$

$$y_2=\frac{564{,}777.4w}{24E}+\frac{w}{24E}\left[\frac{6\times 58^2\times 38^2-4\times 58\times 38^3+38^4}{2.81}\right]$$

$$+\frac{9458.3w\times 12\times 38}{24E}=\frac{11{,}461{,}580.0w}{24E},$$

$$y_1=\frac{11{,}461{,}580.0w}{24E}+\frac{w}{24E}\left[\frac{6\times 20^4-4\times 20^4+20^4}{1.85}\right]$$

$$+\frac{31{,}654.3w\times 12\times 20}{24E}=\frac{19{,}318{,}071.4w}{24E}.$$

When $w=500$ lbs. per ft. as it does in this case,

$$y_1=\varDelta=\frac{19{,}318{,}071.4\times 500}{24\times 144\times 29{,}000{,}000}=.0964 \text{ ft.}$$

ART. 17. THE HORIZONTAL DEFLECTION OF THE UPPER LEFT CORNER OF A SIMPLE BEAM

The previous article shows how the second differential equation of the elastic line may be used to find the deflection of a beam with a varying moment of inertia. The application of the formula $\varDelta=\int_0^l \frac{Mmdx}{EI}$ to the solution of problems in deflection is much simpler than that by using the second differential equation $EI\frac{d^2y}{dx^2}=M$, as for the former the integration is only once performed and between given limits, thus obviating the long and tedious operations necessary to deter-

mine the constants of integration when the latter method is used. In this and several of the following articles the application of the formula of Art. 8 to finding deflections of straight beams will be quite fully illustrated.

As a first illustration of the general application of this formula, suppose it be required to find the horizontal deflection of the upper corner of a simple I-beam span, loaded at the center with the weight P. Fig. 17a will make the problem more clear.

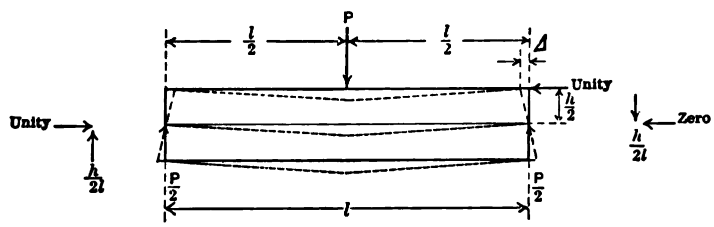


Fig. 17a.

Taking each half of the beam separately, and taking the ends of the neutra axis of the beam as the origin of x, we have:

$$\Delta = \int_0^l \frac{Mmdx}{EI} = \int_0^{\frac{l}{2}} \frac{Mmdx}{EI} \text{ for the left half} + \int_0^{\frac{l}{2}} \frac{Mmdx}{EI} \text{ for the right half,}$$

in which,

$$M = +\frac{P}{2}\times x, \text{ for both right and left halves of the beam,}$$

$$m = +\frac{h}{2l}\times x, \text{ for the left half of the beam,}$$

and

$$m = -\frac{h}{2l}\times x + \frac{h}{2}, \text{ for the right half of the beam.}$$

Substituting these values for M and m in the general equation, we have:

$$\Delta = \int_0^{\frac{l}{2}} \frac{Phx^2dx}{4lEI} - \int_0^{\frac{l}{2}} \frac{Phx^2dx}{4lEI} + \int_0^{\frac{l}{2}} \frac{Phxdx}{4EI} = \int_0^{\frac{l}{2}} \frac{Phxdx}{4EI} = \frac{Phl^2}{32EI}.$$

If we make $P = 29{,}000$, $l = 360$ ins., and $I = 1140$ (ins.$^{4\text{th}}$), this being the value for the moment of inertia of a 20-in. I-beam (192 lbs. per yd.), then:

$$\Delta = \frac{29{,}000\times 20\times(360)^2}{32\times 29{,}000{,}000\times 1140} = \frac{1}{14} \text{ in., nearly.}$$

PROBLEM

No. **17a**. Find the angular rotation of the plane of the left end of the beam of this article due to moment of 200,000 inch-pounds acting in an anticlockwise direction on the left end, when held in equilibrium by an equal and opposite moment at the right end.

ART. 18. THE VERTICAL DEFLECTION AT THE CENTER OF A GIRDER WITH VARYING MOMENT OF INERTIA UNDER UNIFORM LOAD

In the problems of practice it is usually the vertical deflection of some point in a horizontal beam under vertical loading that is desired. The five following articles are deemed sufficient to show the application of the general formula to finding such deflections. The girders in the following problems have four values for the moment of inertia; the resulting expressions for the deflections, however, are of such form that expressions for the value of the deflection for a girder with a greater or less number of values of the moment of inertia can be written by inspection. For girders with inclined flanges, values of the moment of inertia can be determined at stated intervals, and these values considered constant for the intervals taken.

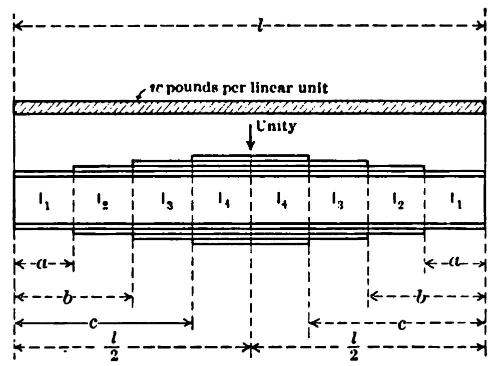


FIG. 18a.

Let it be required to find the vertical deflection at the center of a plate-girder span with three cover plates, the girder being loaded uniformly throughout its length, l, with a vertical load of w pounds per unit of length. Fig. 18a will make the problem more clear.

As the moment of inertia of the girder has four values, the second part of the general equation,

$$\Delta = \int_0^l \frac{Mmdx}{EI} = 2\int_0^{\frac{l}{2}} \frac{Mmdx}{EI},$$

will have four parts:

$$\Delta = \frac{w}{2E}\int_0^a \frac{(lx^2-x^3)dx}{I_1} + \frac{w}{2E}\int_a^b \frac{(lx^2-x^3)dx}{I_2} + \frac{w}{2E}\int_b^c \frac{(lx^2-x^3)dx}{I_2} + \frac{w}{2E}\int_c^{\frac{l}{2}} \frac{(lx^2-x^3)dx}{I_4}$$

$$= \frac{w}{2E}\left\{\frac{la^3}{3I_1} - \frac{a^4}{4I_1} + l\frac{(b^3-a^3)}{3I_2} - \frac{b^4-a^4}{4I_2} + l\frac{(c^3-b^3)}{3I_3} - \frac{c^4-b^4}{4I_3} + \frac{l(l^3-8c^3)}{24I_4} - \frac{l^4-16c^4}{64I_4}\right\}$$

$$= \frac{5}{384}\frac{wl^4}{EI_4} + \frac{wl}{6E}\left(\frac{a^3}{I_1} - \frac{a^3}{I_2} + \frac{b^3}{I_2} - \frac{b^3}{I_3} + \frac{c^3}{I_3} - \frac{c^3}{I_4}\right) - \frac{w}{8E}\left(\frac{a^4}{I_1} - \frac{a^4}{I_2} + \frac{b^4}{I_2} - \frac{b^4}{I_3} + \frac{c^4}{I_4} - \frac{c^4}{I_4}\right),$$

which is the expression for the value of the desired deflection.

If $I_1 = I_2 = I_3 = I_4$, that is, if the moment of inertia of the beam be constant, the expression becomes:

$$\Delta = \frac{5wl^4}{384EI},$$

which is the well-known formula for the deflection at the center of a simple beam of constant moment of inertia, under uniform loading.

ART. 19. THE VERTICAL DEFLECTION AT THE CENTER OF A GIRDER WITH VARYING MOMENT OF INERTIA UNDER A CONCENTRATED LOAD AT ANY POINT IN THE SPAN

Let it be required to find the vertical deflection at the center of a plate-girder span with three cover-plates, due to a vertical load, P, at any point in the span.

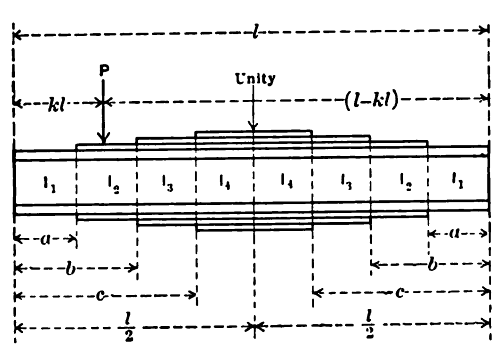


FIG. 19a.

The load P will be placed on the first cover-plate, and the expression found for the value of the deflection. The expression for the deflection for any desired position of P can then be written by inspection.

$$\Delta=\int_0^l \frac{Mmdx}{EI}$$

$$=\int_0^a \frac{P(1-k)x^2dx}{2EI_1}+\int_a^{kl}\frac{P(1-k)x^2dx}{2EI_2}+\int_{kl}^b\frac{Pk(lx-x^2)dx}{2EI_2}+\int_b^c\frac{Pk(lx-x^2)dx}{2EI_3}$$

$$+\int_c^{\frac{l}{2}}\frac{Pk(lx-x^2)dx}{2EI_4}+\int_0^a\frac{Pkx^2dx}{2EI_1}+\int_a^b\frac{Pkx^2dx}{2EI_2}+\int_b^c\frac{Pkx^2dx}{2EI_3}+\int_c^{\frac{l}{2}}\frac{Pkx^2dx}{2EI_4},$$

$$=\frac{P(1-k)}{6E}\left(\frac{a^3}{I_1}-\frac{a^3}{I_2}+\frac{k^3l^3}{I_2}\right)+\frac{Pk}{4E}\left(\frac{lb^2}{I_2}-\frac{lb^2}{I_3}+\frac{lc^2}{I_3}-\frac{lc^2}{I_4}+\frac{l^3}{4I_4}-\frac{l^3k^2}{I_2}\right)$$

$$-\frac{Pk}{6E}\left(\frac{b^3}{I_2}-\frac{b^3}{I_3}+\frac{c^3}{I_3}-\frac{c^3}{I_4}+\frac{l^3}{8I_4}-\frac{k^3l^3}{I_2}\right)+\frac{Pk}{6E}\left(\frac{a_3}{I_1}-\frac{a^3}{I_2}+\frac{b^3}{I_2}-\frac{b^3}{I_3}+\frac{c^3}{I_3}-\frac{c^3}{I_4}+\frac{l^3}{8I_4}\right)$$

$$=\frac{P}{6E}\left(\frac{a^3}{I_1}-\frac{a_3}{I_2}\right)-\frac{Pk^3l^3}{12EI_2}+\frac{Pkl}{4E}\left(\frac{b^2}{I_2}-\frac{b^2}{I_3}+\frac{c^2}{I_3}-\frac{c^2}{I_4}\right)+\frac{Pkl^3}{16EI_4},$$

which is the expression for the value of the desired deflection.

For a load P on the second cover-plate, by inspection, can be written:

$$\Delta=\frac{P}{6E}\left(\frac{a^3}{I_1}-\frac{a^3}{I_2}+\frac{b^3}{I_2}-\frac{b^3}{I_3}\right)-\frac{Pk^3l^3}{12EI_3}+\frac{Pkl}{4E}\left(\frac{c^2}{I_3}-\frac{c^2}{I_4}\right)+\frac{Pkl^3}{16EI_4}.$$

If we make $I_1=I_2=I_3=I_4$, in either of the above values for Δ, we have:

$$\Delta=\frac{Pkl^3}{48EI}(3-4k^2),$$

and if $k=\frac{1}{2}$, that is, for P in the center of the girder,

$$\Delta=\frac{Pl^3}{48EI}.$$

PROBLEM

No. **19a.** Find the vertical deflection at the center for the girder of this article when loaded at the center with a single concentrated load P.

ART. 20. THE VERTICAL DEFLECTION AT THE LOADED POINT OF A GIRDER WITH VARYING MOMENT OF INERTIA

Let it be required to find the vertical deflection at the loaded point of a plate-girder span with three cover-plates, due to a vertical load, P, at any point in the span.

P will be placed on the first cover-plate.

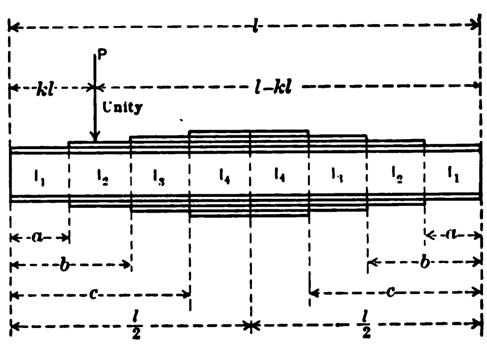


FIG. 20a.

$$\Delta=\int_0^l \frac{Mm\,dx}{EI}$$

$$=\int_0^a \frac{P(1-k)^2x^2dx}{EI_1}+\int_a^{kl}\frac{P(1-k)^2x^2dx}{EI_2}+\int_{kl}^{b}\frac{Pk^2(l-x)^2dx}{EI_2}+\int_b^c\frac{Pk^2(l-x)^2dx}{EI_3}$$

$$+\int_c^{\frac{l}{2}}\frac{Pk^2(l-x)^2dx}{EI_4}+\int_0^a\frac{Pk^2x^2dx}{EI_1}+\int_a^b\frac{Pk^2x^2dx}{EI_2}+\int_b^c\frac{Pk^2x^2dx}{EI_3}+\int_c^{\frac{l}{2}}\frac{Pk^2x^2dx}{EI_4}$$

$$=\frac{P(1-k)^2}{3E}\left(\frac{a^3}{I_1}+\frac{k^3l^3}{I_2}-\frac{a^3}{I_2}\right)+\frac{Pk^2l^2}{E}\left(\frac{b}{I_2}-\frac{kl}{I_2}+\frac{c}{I_3}-\frac{b}{I_3}+\frac{l}{2I_4}-\frac{c}{I_4}\right)$$

$$-\frac{Pk^2l}{E}\left(\frac{b^2}{I_2}-\frac{k^2l^2}{I_2}+\frac{c^2}{I_3}-\frac{b^2}{I_3}+\frac{l^2}{4I_4}-\frac{c^2}{I_4}\right)+\frac{Pk^2}{3E}\left(\frac{b^3}{I_2}-\frac{k^3l^3}{I_2}+\frac{c^3}{I_3}-\frac{b^3}{I_3}+\frac{l^3}{8I_4}-\frac{c^3}{I_4}\right)$$

$$+\frac{Pk^2}{3E}\left(\frac{a^3}{I_1}+\frac{b^3}{I_2}-\frac{a^3}{I_3}+\frac{c^3}{I_3}-\frac{b^3}{I_3}+\frac{l^3}{8I_4}-\frac{c^2}{I_4}\right)$$

$$=\frac{Pl^3k^2}{3E}\left(\frac{1}{I_4}-\frac{2k}{I_2}+\frac{k^2}{I_2}\right)+\frac{P(1-2k)}{3E}\left(\frac{a^3}{I_1}-\frac{a^3}{I_2}\right)+\frac{Pk^2l^2}{E}\left(\frac{b}{I_2}-\frac{b}{I_3}+\frac{c}{I_3}-\frac{c}{I_4}\right)$$

$$-\frac{Pk^2l}{E}\left(\frac{b^2}{I_2}-\frac{b^2}{I_3}+\frac{c^2}{I_3}-\frac{c}{I_4}\right)+\frac{2Pk^2}{3E}\left(\frac{a^3}{I_1}-\frac{a^3}{I_2}+\frac{b^3}{I_2}-\frac{b^3}{I_3}+\frac{c^3}{I_4}-\frac{c^3}{I_4}\right),$$

which is the expression for the value of the desired deflection.

If we make $I_1 = I_2 = I_3 = I_4$, then

$$\Delta = \frac{Pl^3k^2(1-k)^2}{3EI},$$

and, in addition, if we make $k = \frac{1}{2}$, then

$$\Delta = \frac{Pl^3}{48EI}.$$

ART. 21. THE VERTICAL DEFLECTION OF THE END OF A CANTILEVER GIRDER OF VARYING MOMENT OF INERTIA UNDER UNIFORM LOAD

Let it be required to find the vertical deflections of the ends of a plate-girder span with three cover-plates, supported at the center only, and loaded with a uniform vertical load, w, per unit of length.

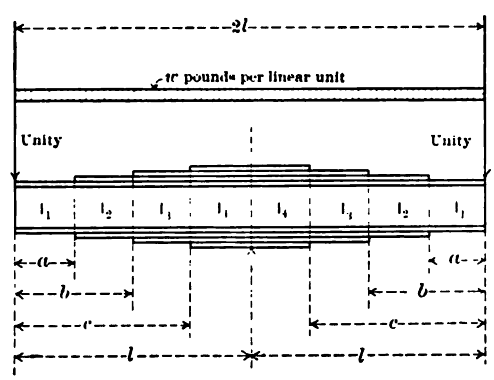


FIG. 21*a*.

$$\Delta = \int_0^l \frac{Mmdx}{EI}$$

$$= \int_0^a \frac{wx^3dx}{2EI_1} + \int_a^b \frac{wx^3dx}{2EI_2} + \int_b^c \frac{wx^3dx}{2EI_3} + \int_c^l \frac{wx^3dx}{2EI_4}$$

$$= \frac{w}{8E}\left(\frac{a^4}{I_1} + \frac{b^4}{I_2} - \frac{a^4}{I_2} + \frac{c^4}{I_3} - \frac{b^4}{I_3} + \frac{l^4}{I_4} - \frac{c^4}{I_4}\right)$$

$$= \frac{wl^4}{8EI_4} + \frac{w}{8E}\left(\frac{a^4}{I_1} - \frac{a^4}{I_2} + \frac{b^4}{I_2} - \frac{b^4}{I_3} + \frac{c^4}{I_3} - \frac{c^4}{I_4}\right),$$

which is the expression for the value of the desired deflection.

If $I_1 = I_2 = I_3 = I_4$, then,

$$\Delta = \frac{wl^4}{8EI}.$$

ART. 22. THE VERTICAL DEFLECTION AT THE END OF A CANTILEVER GIRDER OF VARYING MOMENT OF INERTIA FOR A LOAD AT THE END

Let it be required to find the vertical deflection of the ends of a plate-girder span with three cover-plates, supported at the center only, and loaded at each end with a load, P.

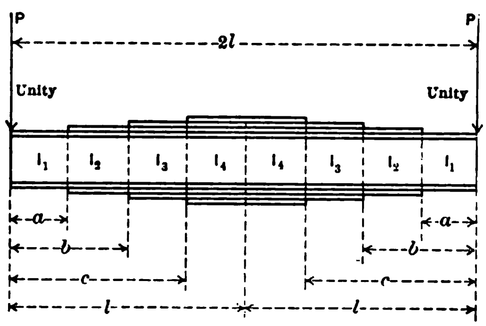


FIG. 22a.

$$\varDelta=\int_0^l\frac{Mmdx}{EI}$$

$$=\int_0^a\frac{Px^2dx}{EI_1}+\int_a^b\frac{Px^2dx}{EI_2}+\int_b^c\frac{Px^2dx}{EI_3}+\int_c^l\frac{Px^2dx}{EI_4}$$

$$=\frac{P}{3E}\left(\frac{a^3}{I_1}+\frac{b^3}{I_2}-\frac{a^3}{I_2}+\frac{c^3}{I_3}-\frac{b^3}{I_3}+\frac{l^3}{I_4}-\frac{c^3}{I_4}\right)$$

$$=\frac{Pl^3}{3EI_4}+\frac{P}{3E}\left(\frac{a^3}{I_1}-\frac{a^3}{I_2}+\frac{b^3}{I_2}-\frac{b^3}{I_3}+\frac{c^3}{I_3}-\frac{c^3}{I_4}\right),$$

which is the expression for the value of the desired deflection.

If $I_1=I_2=I_3=I_4$, then,

$$\varDelta=\frac{Pl_3}{3EI}.$$

By substituting for $\varDelta$, in the above expression, the distance it is desired to raise the end of a plate-girder draw, and solving for P, the uplift necessary to raise the end the desired amount is obtained.

In applying the results of the preceding problems, a deduction of 3 or 4 ft. from the over-all lengths of the cover-plates should be made, for the reason that

enough rivets to fully develop the value of a cover-plate to resist stress are not generally contained in a less distance than the first 1½ or 2 ft. at the end of the cover plate.

ART. 23. TRUE REACTIONS FOR PLATE-GIRDER SWING BRIDGES

By the aid of the results of Arts. 21 and 22 the reactions for center-bearing plate-girder draw spans, with equal arms under uniform loading, can be obtained.

Let $R_1 = R_3$ be the reactions at the ends of a two-span plate-girder with three cover-plates. From the result of Art. 22, we have, for the vertical motion, Δ, due to a force, $R_1 = R_3$, at the ends, the following:

$$\Delta = \frac{R_1 l^3}{3EI_4} + \frac{R_1}{3E}\left(\frac{a^3}{I_1} - \frac{a^3}{I_2} + \frac{b^3}{I_2} - \frac{b^3}{I_3} + \frac{c^3}{I_3} - \frac{c^3}{I_4}\right).$$

If, at the same time, we make the value for Δ given by the result of Art. 21 equal to the value for Δ above, and solve the equation for R_1, we have:

$$R_1 = R_3 = \frac{\dfrac{wl^4}{8I_3} + \dfrac{w}{8}\left(\dfrac{a^4}{I_1} - \dfrac{a^4}{I_2} + \dfrac{b^4}{I_2} - \dfrac{b^4}{I_3} + \dfrac{c^4}{I_3} - \dfrac{c^4}{I_4}\right)}{\dfrac{l^3}{3I_4} + \dfrac{1}{3}\left(\dfrac{a^3}{I_1} - \dfrac{a^3}{I_2} + \dfrac{b^3}{I_2} - \dfrac{b^3}{I_3} + \dfrac{c^3}{I_3} - \dfrac{c^3}{I_4}\right)} = \frac{3w}{8}\,\frac{\dfrac{l^4}{I_4} + \dfrac{a^4}{I_1} - \dfrac{a^4}{I_2} + \dfrac{b^4}{I_2} - \dfrac{b^4}{I_3} + \dfrac{c^4}{I_3} - \dfrac{c^4}{I_4}}{\dfrac{l^3}{I_4} + \dfrac{a^3}{I_1} - \dfrac{a^3}{I_2} + \dfrac{b^3}{I_2} - \dfrac{b^3}{I_3} + \dfrac{c^3}{I_3} - \dfrac{c^3}{I_4}},$$

which is the expression for the value of the end reactions for a plate-girder of two equal spans having four values for the moment of inertia, and loaded uniformly throughout its length by w pounds per unit of length.

If $I_1 = I_2 = I_3 = I_4$, then $R_1 = R_3 = \frac{3}{8}wl$.

It would not be a difficult matter to derive values for the reactions for a more general case of loading and span lengths; but the ordinary plate-girder draw has equal arms, and the only place, under the general practice in designing, where it is necessary to consider the effect of continuous loading is near and over the center support. The maximum values for shear and moment at the center support are given under conditions closely approximating continuous uniform load, and this is the problem which has been solved.

ART. 24. COMPARISON OF METHODS FOR COMPUTING SWING-BRIDGE REACTIONS

In order to see how much the ordinary method (using a constant moment of inertia) of computing the reactions and the moments over the center supports of plate-girder swing bridges differs from this more exact method, the comparison will be made from two widely different cases.

Case I.—A very light, single-track, center-bearing, plate-girder draw, 72½ inches deep, from out to out of flange-angles.

		Length over All.		Effective Length.
1 web plate,	72×⅜ ins.	138 ft.	and	136 ft.
4 flange-angles, (55 lbs. per yard.)	5×5 "	138 "	"	136 "
2 cover-plates,	12×⅝ "	98 "	"	96 "
2 " "	12×⅝ "	22 "	"	20 "
2 " "	12×⅝ "	10 "	"	8 "

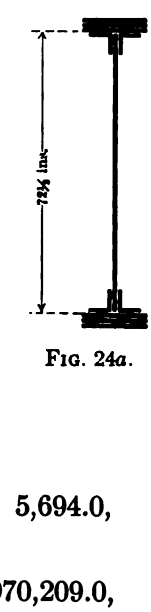


FIG. 24a.

$I_1 =$ 38,248 ins.^{4} $=1.85$ ft.^{4} and $a=20$ ft.

$I_2 =$ 58,335 " $=2.81$ " $b=58$ "

$I_3 =$ 79,081 " $=3.81$ " $c=64$ "

$I_4 =$ 100,526 " $=4.85$ " $l=68$ "

$\frac{a^3}{I_1}=$ 4,324.3,	$\frac{a^3}{I_2}=$ 2.847.0,	$\frac{a^4}{I_1}=$ 8,648.6,	$\frac{a^4}{I_2}=$ 5,694.0,
$\frac{b^3}{I_2}=$ 69,434.9,	$\frac{b^3}{I_3}=$ 51,210.5,	$\frac{b^4}{I_2}=$ 4,027,206.3,	$\frac{b^4}{I_3}=$ 2,970,209.0,
$\frac{c^3}{I_3}=$ 68,804.2,	$\frac{c^3}{I_4}=$ 54,050.3,	$\frac{c^4}{I_3}=$ 4,403,468.8,	$\frac{c^4}{I_4}=$ 3,459,219.8,
$\frac{l^3}{I_4}=$ 64,831.3,		$\frac{l^4}{I_4}=$ 4,408,531.1,	
$+ =$ 207,394.7	$- =$ 108,107.8	$+ =$ 12,847,854.8	$- =$ 6,435,122.8

$$R_1=\frac{3}{8}w\frac{\frac{l^4}{I_4}+\frac{a^4}{I_1}-\frac{a^4}{I_2}+\frac{b^4}{I_2}-\frac{b^4}{I_3}+\frac{c^4}{I_3}-\frac{c^4}{I_4}}{\frac{l^3}{I_4}+\frac{a^3}{I_1}-\frac{a^3}{I_3}+\frac{b^3}{I_3}-\frac{b^4}{I_3}+\frac{c^4}{I_3}-\frac{c^4}{I_4}}$$

$$=\frac{3}{8}w\frac{12{,}847{,}854.8-6{,}435{,}132.8}{207{,}394.7-108{,}107.8}=\frac{3}{8}w64.59=24.22w.$$

The moment over the center support for this case is:

$$M = 68w(24.22 - 34.00) = -665.04w \text{ ft.-lbs.}$$

The moment over the center support for constant moment of inertia is:

$$M = -\tfrac{1}{8}wl^2 = -578w \text{ ft.-lbs,}$$

which shows that this more exact method gives a moment over the center support for this case 15 per cent greater than the usual method.

Case II.—A very heavy, double-track, center-bearing, plate-girder draw, $101\frac{1}{2}$ inches deep, from out to out of flange angles.

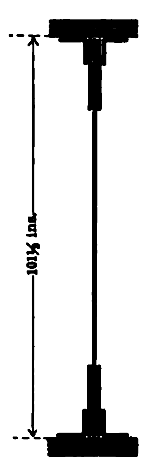


FIG. 24b.

			Length over All.		Effective Length.
1 web plate,	$101 \times \frac{11}{16}$	ins.,	159 ft.	and	156 ft.
4 side flange-plates,	$18 \times \frac{3}{4}$	"	159 "	"	156 "
4 flange angles, (125 lbs. per yard.)	8×6	"	159 "	"	156 "
2 cover-plates,	$20 \times \frac{3}{4}$	"	159 "	"	156 "
2 " "	$20 \times \frac{3}{4}$	"	84 "	"	82 "
2 " "	$20 \times \frac{3}{4}$	"	24 "	"	22 "
2 " "	$20 \times \frac{3}{4}$	"	16 "	"	14 "

$I_1 = 353{,}750$ ins.^{4} $= 17.06$ ft.^{4} and $a = 37$ ft.
$I_2 = 434{,}500$ " $= 20.96$ " " $b = 67$ "
$I_3 = 517{,}600$ " $= 24.96$ " " $c = 71$ "
$I_4 = 603{,}100$ " $= 29.08$ " " $l = 78$ "

$\frac{a^3}{I_1} = 2{,}969.1,$	$\frac{a^3}{I_2} = 2{,}416.7,$	$\frac{a^4}{I_1} = 109{,}857.3,$	$\frac{a^4}{I_2} = 89{,}416.1,$
$\frac{b^3}{I_2} = 14{,}349.4,$	$\frac{b^3}{I_3} = 12{,}049.9,$	$\frac{b^4}{I_2} = 961{,}408.5,$	$\frac{b^4}{I_3} = 807{,}336.6,$
$\frac{c^3}{I_3} = 14{,}339.3,$	$\frac{c^3}{I_4} = 12{,}307.8,$	$\frac{c^4}{I_3} = 1{,}018{,}096.2,$	$\frac{c^4}{I_4} = 873{,}854.2,$
$\frac{l^3}{I_4} = 16{,}318.8,$		$\frac{l^4}{I_4} = 1{,}272{,}869.9,$	
$+ = 47{,}976.6$	$- = 26{,}774.4$	$+ = 3{,}362{,}231.9$	$- = 1{,}770{,}606.9$

$$R_1 = \frac{3}{8}w\frac{3{,}362{,}231.9 - 1{,}770{,}606.9}{47{,}976.6 - 26{,}774.4} = \frac{3}{8}w75.07 = 28.15w.$$

The moment over the center support, for this case, is:

$$M = 78w(28.15 - 39.00) = -846.30w \text{ ft.-lbs.}$$

The moment over the center support, for constant moment of inertia, is:

$$M = -\tfrac{1}{8}wl^2 = 760.5w \text{ ft.-lbs.}$$

Which shows that this more exact method gives a moment over the center support for this case 11 per cent greater than the usual method.

The two plate-girder drawbridges selected for investigation may be said to represent fairly the results obtained by the method commonly used in designing such structures.

ART. 25. SHEARING STRESSES IN PRODUCING DEFLECTION FOR PLATE GIRDERS

In order to determine the effect of the shearing stresses in producing deflection, the following investigation is made:

Let there be taken any portion of a beam in equilibrium under the action of transverse loading, as shown in Fig. 25*a*. If P be the resultant in position, direction and amount of all the forces to the right of the section mn, then it is clear that the portion of the beam under consideration would be kept in equilibrium by the couple, Fa, and the shear, S.

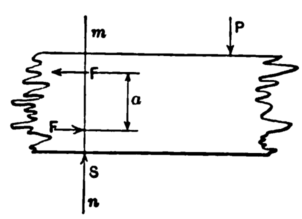


FIG. 25*a*.

In general, no matter what may be the actual distribution of the stresses on any section of a beam under a given loading, these stresses may be resolved into an equivalent couple and shear. The effect of the couple in doing work has been determined, in Art. 8, in accordance with the common theory of flexure. Going back to Art. 8, and rewriting the expression for the work of the internal stresses, we have for the total internal work

$$= w = \left(\int_0^l \frac{Mmdx}{2EI} \text{ for the work of the flexural stresses}\right)$$

$$+ \left(\int_0^l \frac{Ssdx}{2E_sA} \text{ for the work of the shearing stresses}\right).$$

To derive this expression for the work done by the shearing stresses:

Let S = shear at any point, due to the given loading;

s = shear at the same point, due to a load of unity acting at the point where deflection is desired;

A = total area of the cross-section;

E_s = coefficient of elasticity for shear;

and let dx be an infinitesimal portion of the length of the beam; that is, let it be considered as the distance between two consecutive sections of the beam.

Then, under the action of the shearing stress, S, the two consecutive sections will have a relative motion of $\frac{Sdx}{AE_s}$, if we assume that the shearing stress is distributed uniformly over the cross-section.

Now, as for the flexural stresses, we will suppose the force of unity to be applied gradually; then the work done by the shearing stress, on an elementary portion of the length of the beam, due to the force of unity, is $\frac{Sdx}{AE_s} \times \frac{s}{2}$, and, over the entire length of the beam, the work is $\int_0^l \frac{Ssdx}{2E_sA}$.

Making the expressions for the external and internal work equal, we have:

$$\frac{\Delta}{2} = \int_0^l \frac{Mmdx}{2EI} + \int_0^l \frac{Ssdx}{2E_sA},$$

and

$$\Delta = \int_0^l \frac{Mmdx}{EI} + \int_0^l \frac{Ssdx}{E_sA}.$$

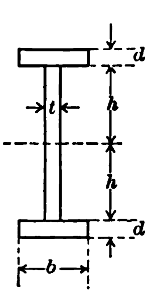


FIG. 25b.

The shearing stress is not distributed uniformly over the cross-section, and hence the term for the deflection due to the shearing stress should be modified by a coefficient. The value of this coefficient for a rectangular section has been shown in Art. 9 to be $\frac{6}{5}$.

For a section, as shown in Fig. 25b, which is a close enough approximation to plate-girder cross-sections, for all practical purposes, the deflection caused by the shearing stresses on two flanges

$$= \frac{b[8(h+d)^5 - 15(h+d)^4h + 10(h+d)^2h^3 - 3h^5]}{30I^2} \int_0^l \frac{Ssdx}{E_s},$$

and on the web

$$= \frac{60b^2d^2\frac{h}{t}\left(h+\frac{d}{2}\right)^2 + 40bdh^3\left(h+\frac{d}{2}\right) + 8h^5t}{30I^2} \int_0^l \frac{Ssdx}{E_s}.$$

The total deflection for the shearing stresses on the entire cross-section is the sum of these, and

$$=\left[8b(h+d)^5-15bh(h+d)^4+10bh^3(h+d)^2-3bh^5+60b^2d^2\frac{h}{t}\left(h+\frac{d}{2}\right)^2\right.$$
$$\left.+40bdh^3\left(h+\frac{d}{2}\right)+8h^5t\right]\int_0^l \frac{Ssdx}{30I^2E_s}.$$

This expression is of such an involved nature that a simple inspection of the quantity before the integral sign gives very little information as to the relative amount of the deflection due to the shear on the flanges or web to that of the deflection due to the shear on the entire cross-section. For the purpose of getting a definite idea of this relation, and, at the same time, for reinvestigating the plate girders of Case I and Case II of Art. 24, approximate sections for the end and center of each of these girders will be assumed, and the actual value of the quantity before the integral sign divided by $30I^2$, which will be called C, computed.

Case I.

Approximate end-section:	Approximate center-section:
$b=12$ ins.	$b=12$ ins.
$d=1$ "	$d=2$ "
$h=36$ "	$h=36$ "
$t=\frac{3}{8}$ "	$t=\frac{3}{8}$ "
Actual coefficient $C=.036+$	$.037+$
$\frac{1}{\text{area of web}}=\frac{1}{27}=.037+$	$.037+$

Case II.

Approximate end-section:	Approximate center-section:
$b=20$ ins.	$b=20$ ins.
$d=2$ "	$d=4$ "
$h=50$ "	$h=50$ "
$t=\frac{3}{4}$ "	$t=\frac{3}{4}$ "
Actual coefficient $C=.013+$	$.013+$
$\frac{1}{\text{area of web}}=\frac{1}{75}=.013-$	$.013-$

From this it is seen that the amount of deflection due to the shear on the flanges is a very small part of the deflection due to shear on the entire cross-section. A similar investigation on a section more closely approximating the actual cross-section of a plate girder, as shown in Fig. 25c, would show a considerably larger amount of deflection due to the flanges; but it would still be very small. Therefore, in investigating the effect of shear in producing distortion on plate girders it will be very nearly correct to assume that the web alone resists all the shearing stresses and that the shear is uniformly distributed. This assumption will give deflections a little too great, as the flanges of a plate girder do take a small-part of the shear—for a plate girder such as that in Case II the vertical flange plates and the vertical legs of the flange angles take considerable shear.

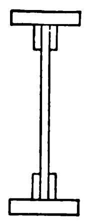


FIG. 25c.

The general equation for the deflection of a plate girder of varying cross-section now becomes

$$\Delta = \int_0^l \frac{Mmdx}{EI} + \int_0^l \frac{Ssdx}{E_s A_w},$$

in which Aw = area of web.

Making $E_s = \frac{2}{5}E$, the deflection

$$\Delta = \int_0^l \frac{Mmdx}{EI} + \frac{5}{2}\int_0^l \frac{Ssdx}{EA_w}.$$

From this the expression for the value of the deflection for the Problem of Art. 21, becomes

$$\Delta = \frac{w}{8E}\left(\frac{l^4}{I_4} + \frac{a^4}{I_1} - \frac{a^4}{I_2} + \frac{b^4}{I_2} - \frac{b^4}{I_3} + \frac{c^4}{I_3} - \frac{c^4}{I_4}\right) + \frac{5}{4}\frac{wl^2}{EA_w};$$

and for the Problem of Art. 22 it becomes

$$\Delta = \frac{P}{3E}\left(\frac{l^3}{I_4} + \frac{a^3}{I_1} - \frac{a^3}{I_2} + \frac{b^3}{I_2} - \frac{b^3}{I_3} + \frac{c^3}{I_3} - \frac{c^3}{I_4}\right) + \frac{5}{2}\frac{Pl}{EA_w}.$$

The expression for the value of the end reactions for a plate girder of two equal spans, the girder having four values of I, and loaded uniformly throughout its length by w pounds per unit of length, is:

$$R_1 = R_3 = \frac{3w}{8} \cdot \frac{\left(\frac{l}{I_4} + \frac{a^4}{I_1} - \frac{a^4}{I_2} + \frac{b^4}{I_2} - \frac{b^4}{I_3} + \frac{c^4}{I_3} - \frac{c^4}{I_4}\right) + \frac{10l^2}{A_w}}{\left(\frac{l^3}{I_4} + \frac{a^3}{I_1} - \frac{a^3}{I_2} + \frac{b^4}{I_2} - \frac{b^3}{I_3} + \frac{c^3}{I_3} - \frac{c^3}{I_4}\right) + \frac{15l}{2A_w}}.$$

Using this expression in investigating Case I and Case II:

For Case I:

$$R = \frac{3}{8}w\frac{6{,}412{,}722 + 246{,}613}{99{,}287 + 2{,}720} = \frac{3}{8}w \times 65.28 = 24.48w;$$

and the moment over the center support,

$$M = 68w(24.48 - 34.00) = -647.36w \text{ ft.-lbs.}$$

This moment is only 2.6 per cent less than that found by the first investigation for Case I.

For Case II:

$$R = \frac{3}{8}w\frac{1{,}591{,}625+116{,}813}{21{,}202+1123} = \frac{3}{8}w \times 76.52 = 28.70w;$$

and the moment over the center support,

$$M = 78w(28.70 - 39.00) = -803.4w \text{ ft.-lbs.}$$

This moment is 5 per cent less than that found by the first investigation for Case II.

The result of this reinvestigation for Case I and Case II shows: For the shorter and lighter draw-span, that the deflection of the end of the span, due to the shearing stresses caused by uniform loading, is about 4 per cent of that due to the flexural stresses from the same cause; that the deflection of the end of the span due to the shear caused by a load at the end is about 3 per cent of that due to the flexure stresses from the same cause; and that the calculated moment over the center support, obtained by a consideration of the flexural stresses alone, is about 2.6 per cent greater than that obtained by taking into account the effect of the shearing stresses.

For the longer and heavier draw-span, the consideration of the effect of the shearing stresses shows differences about twice as great, in each instance, as was shown for the shorter and lighter structure.

That the effect of the shearing stresses should be greatest in the long girder, does not seem to be in accord with the fact that, for a beam of constant cross-section, the relative effect of the shearing stresses is less the longer the span; an inspection of all the elements of the problem shows that the much larger moment of inertia and the relatively less area in the web of the long girder are more than sufficient to overcome the effect of its greater length. In view of the results of this investigation it would seem to be necessary to consider the effect of the shearing stresses, in modifying the effect of the flexural stresses in designing, only for very deep girders subject to very heavy loads. The amount of labor required for a proper consideration of the shearing stresses is very small, so that whenever great accuracy is required they may be considered.

ART. 26. SPECIAL PROBLEM BY MEANS OF THE FRAENKEL FORMULA

The example of Art. 16 will now be solved by the general equation

$$\Delta = \int \frac{Mmdx}{EI},$$

$$\Delta = \frac{wL^4}{8EI} + \frac{w}{8E}\left[\frac{c^4}{I_1} - \frac{c^4}{I_2} + \frac{b^4}{I_2} - \frac{b^4}{I_3} + \frac{a^4}{I_3} - \frac{a^4}{I_4}\right] - \frac{wL}{2E}\left[\frac{c^3}{I_1} - \frac{c^3}{I_2} + \frac{b^3}{I_2} - \frac{b^3}{I_3} + \frac{a^3}{I_3} - \frac{a^3}{I_4}\right]$$

$$+ \frac{3wL^2}{4E}\left[\frac{c^2}{I_1} - \frac{c^2}{I_2} + \frac{b^2}{I_2} - \frac{b^2}{I_3} + \frac{a^2}{I_3} - \frac{a^2}{I_4}\right] - \frac{wL^3}{2E}\left[\frac{c}{I_1} - \frac{c}{I_2} + \frac{b}{I_2} - \frac{b}{I_3} + \frac{a}{I_3} - \frac{a}{I_4}\right],$$

$$\Delta = \frac{1{,}444{,}688.8w}{E} + \frac{123{,}905.8w}{E} - \frac{697{,}676.5w}{E} + \frac{1{,}511{,}354.4w}{E} - \frac{1{,}576{,}876.5w}{E},$$

$$\Delta = \frac{805{,}396 \times 500}{144 \times 29{,}000{,}000} = .0965 \text{ ft.},$$

which gives almost exactly the same result as the former method with not more than one-tenth the labor.

ART. 27. DETERMINATION OF DEFLECTIONS BY MEANS OF APPROXIMATE METHODS OF INTEGRATION

For many structures subject to flexure the elastic line is a curve or series of curves which may not be represented by any known equation. In such cases $\int_0^l \frac{Mmdx}{EI}$ cannot be determined except by an approximation. For such cases it is

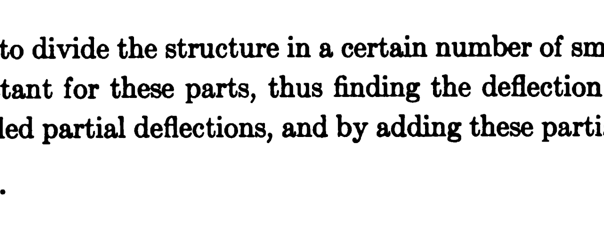


FIG. 27a.

sufficiently exact to divide the structure in a certain number of small parts and take M and m as constant for these parts, thus finding the deflection due to the parts, which may be called partial deflections, and by adding these partial deflections thus obtain $\int_0^l \frac{Mmdx}{EI}$.

To show the degree of accuracy of the method take a case where $\int_0^l \frac{Mmdx}{EI}$ can readily be integrated by the calculus and compare the result with the approximate method.

To illustrate, take a 12-inch steel I—31.5 lbs. projecting 10 ft. from a wall and loaded with 10,000 lbs. at the end as illustrated in Fig. 27*a*. Suppose the beam to be divided into 10 equal parts each 1 ft. long. Then $\frac{Mmdx}{EI}$ for each part is as follows:

$$\text{for No. 1,} \quad \frac{Mmdx}{EI}=\frac{60{,}000\times 6\times 12}{EI};$$

$$\text{“} \quad 2, \quad \frac{Mmdx}{EI}=\frac{180{,}000\times 18\times 12}{EI};$$

$$\text{“} \quad 3, \quad \frac{Mmdx}{EI}=\frac{300{,}000\times 30\times 12}{EI};$$

$$\text{“} \quad 4, \quad \frac{Mmdx}{EI}=\frac{420{,}000\times 42\times 12}{EI};$$

$$\text{“} \quad 5, \quad \frac{Mmdx}{EI}=\frac{540{,}000\times 54\times 12}{EI};$$

$$\text{“} \quad 6, \quad \frac{Mmdx}{EI}=\frac{660{,}000\times 66\times 12}{EI};$$

$$\text{“} \quad 7, \quad \frac{Mmdx}{EI}=\frac{780{,}000\times 78\times 12}{EI};$$

$$\text{“} \quad 8, \quad \frac{Mmdx}{EI}=\frac{900{,}000\times 90\times 12}{EI};$$

$$\text{“} \quad 9, \quad \frac{Mmdx}{EI}=\frac{1{,}020{,}000\times 102\times 12}{EI};$$

$$\text{“} \quad 10, \quad \frac{Mmdx}{EI}=\frac{1{,}140{,}000\times 114\times 12}{EI}.$$

For a case where I varies it should be used as a constant over the length of each portion. For this case I is constant, so we may write:

$$\begin{aligned}
\frac{Mmdx}{EI}=\frac{12}{EI}(60{,}000\times\ 6 &= (+\ \ 360{,}000\\
+\ 180{,}000\times 18 &= +\ \ 3{,}240{,}000\\
+\ 300{,}000\times 30 &= +\ \ 9{,}000{,}000\\
+\ 420{,}000\times 42 &= +\ 17{,}640{,}000
\end{aligned}$$

$$
\begin{aligned}
+\ \ 540{,}000\times\ 54 &= +\ \ 29{,}160{,}000\\
+\ \ 660{,}000\times\ 66 &= +\ \ 43{,}560{,}000\\
+\ \ 780{,}000\times\ 78 &= +\ \ 60{,}840{,}000\\
+\ \ 900{,}000\times\ 90 &= +\ \ 81{,}000{,}000\\
+1{,}020{,}000\times 102 &= +104{,}040{,}000\\
+1{,}140{,}000\times 114) &= +129{,}960{,}000)\frac{12}{EI}\\
&= \quad 478{,}800{,}000\times\frac{12}{EI}\\
&= \frac{5{,}745{,}600{,}000}{EI}
\end{aligned}
$$

The value of $\int_0^l \frac{Mmdx}{EI}$ for this well-known case is

$$\frac{Pl^3}{3EI}=\frac{10{,}000\times 120\times 120\times 120}{3}=\frac{5{,}760{,}000{,}000}{EI},$$

which shows that the approximate method has a high degree of accuracy where the divisions are taken small.

PROBLEM

No. **27*a***. Find the deflection of the beam of Fig. 27*a* by the approximate method and taking each division 24 ins. long.

CHAPTER III

DEFLECTIONS AND STRESSES FOR CURVED STRUCTURES WITH SOLID WEBS

ART 28. DEFLECTIONS OF CURVED BEAMS

THE determination of the deflections of a curved bar subject to flexure cannot be made with the same degree of precision as for straight beams. The lengths of the different fibers for any small portion of the beam are not of the same length as the corresponding portion of the axis.

Experience and experiment on the deflection of such bars subject to flexure shows, however, that very accurate theoretical determinations of the deflections may be made by considering all the fibers of the beam to be of the same length

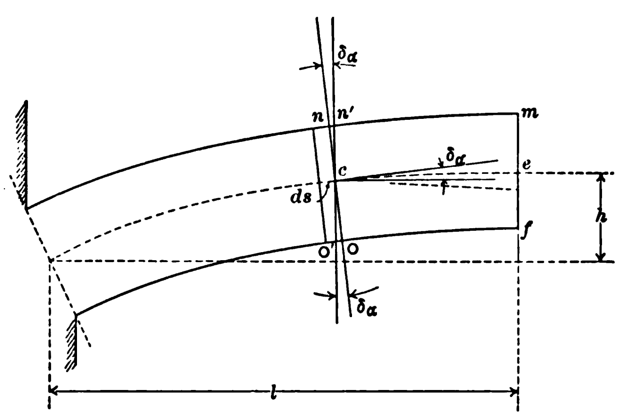


FIG. 28a.

and equal to a corresponding length of the axis. Let Fig. 28a represent any curved beam whose deflection is required, subject to a bending moment M at any cross-section. And let m be the bending moment at the same cross-section for which M is taken and due to a load of unity acting in the direction in which the deflection is desired. Then $\Delta = \int \frac{Mmds}{EI}$ just as was shown in Art. 8 for the straight beam.

The horizontal deflection of e due to any given loading will be

$$\Delta_x = \int \frac{Mm_h ds}{EI},$$

in which M = bending moment due to the given loading on an element of the length of the beam; and m_h = bending moment due to a horizontal force of unity applied at e on the same element for which M is taken. The other quantities do not need definition.

The vertical deflection of e due to the given loading will be

$$\Delta_y = \int \frac{Mm_v ds}{EI},$$

in which M is as before defined and m_v = the bending moment due to a vertical force of unity at e for the same section for which M is taken.

The angular motion of the plane of the end of the beam

$$\Delta_\alpha = \int \frac{Mds}{EI},$$

in which M is as before.

For let M = bending moment due to the given loading on any element of the length of the beam ds, which causes a change in length of the upper fibers of this element of the beam of nn', and of the lower fibers of oo'.

Then the angular motion δ_α of the plane no due to the moment M over the length ds

$$= \frac{oo'}{co} = \frac{nn'}{cn},$$

but

$$oo' = \frac{M.\overline{co}.ds}{IE} \quad \text{and} \quad nn' = \frac{M.\overline{cn}.ds}{EI}.$$

$$\therefore \quad \delta_\alpha = \frac{M.\overline{co}.ds}{EI.\overline{co}} = \frac{M.\overline{cn}.ds}{EI.\overline{cn}} = \frac{Mds}{EI}.$$

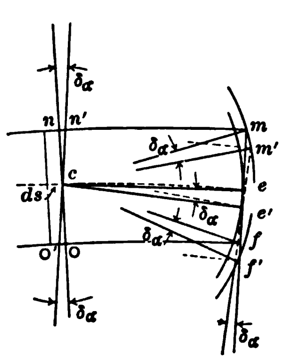


FIG. 28b.

The rotation of the plane no due to the moment M on a length ds produces an equal rotation of the end plane fm of the beam, as can be readily seen from Fig. 28b.

Therefore the total angular rotation of $fm = \Delta_\alpha = \int \frac{Mds}{EI}$ for the entire length of the beam axis.

The vertical and angular deflections are direct functions of and proportional to the forces and moments that produce them.

PROBLEM

No. 28a. Let Fig. 28a represent a curved steel beam of rectangular cross-section 6″×4″, the axis of which is a parabola with the apex at e, l=10 ft., h=4 ft. If this beam be loaded with 100 lbs. p.l.f., compute Δ_x, Δ_y, and Δ_α by the approximate method indicated in Art. 27.

ART. 29. THE DEFLECTIONS OF CIRCULAR RING UNDER TWO EQUAL AND OPPOSITE FORCES

Let it be required to find the stresses in a circular ring under the action of two oppositely directed forces P, the ring being made from a round steel rod $1\frac{3}{8}$ in. in diameter and to diameter of ring of 8 ins. c. to c. all as indicated in Fig. 29*a*. The stresses in the ring at the plane *fn* or any other plane, when it is acting as a whole, are a function of certain deflections which this plane undergoes, when the ring is considered cut, as will be shown later. With the stresses at one point of the ring known they may readily be found at any other point. As a first step, consider the ring to be cut by a horizontal plane *fn* through *b*, and that P at *b* is also divided into two equal parts. The stresses on a cross-section, taken at any point of the ring axis, may now be found by the methods of statics, and the horizontal and vertical motions of *b*, together with the angular motion of plane *fn* under the given loading, may be determined.

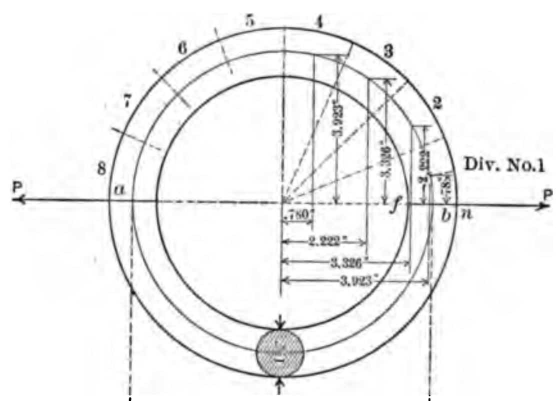


Fig. 29*a*

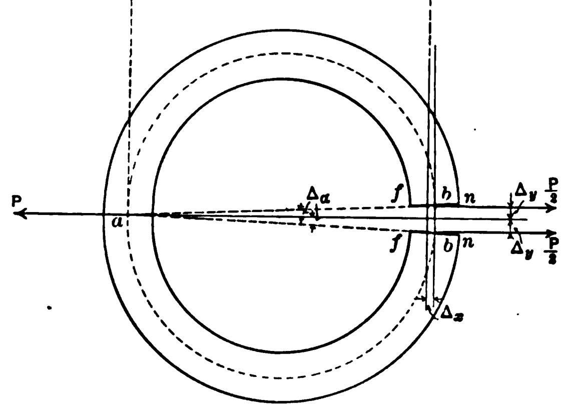


Fig. 29*b*

All motions will be considered to take place about the point *a*.

Let Δ_y = the vertical deflection of *b*;

Δ_x = the horizontal deflection of *b*;

Δ_a = the angular motion of the plane *fn*.

Fig. 29*b* will help in making the definitions for the foregoing deflections more clear.

The computation for these deflections will be arranged as would be required were the ring not circular and of constant cross-section.

TABLE No. 29a

Div. No.	x	y	$\frac{ds}{EI}$	M	$m_h(\overset{1}{\longrightarrow})$	Mm_h	$m_v(1\uparrow)$	Mm_v
1	0.077	0.780	$\frac{1.571}{30000000\times0.175}$	+0.390P	+0.780	+0.304P	+0.077	+0.030P
2	0.674	2.222	do. = 0.000000299	1.111P	2.222	2.468P	0.674	0.749P
3	1.778	3.326	do. = do.	1.663P	3.326	5.532P	1.778	2.957P
4	3.220	3.923	do. = do.	1.961P	3.923	7.694P	3.220	6.314P
5	4.780	3.923	do. = do.	1.962P	3.923	7.694P	4.780	9.378P
6	6.222	3.326	do. = do.	1.663P	3.326	5.532P	6.222	10.347P
7	7.326	2.222	do. = do.	1.111P	2.222	2.468P	7.326	8.139P
8	7.923	0.780	do. = do.	0.390P	0.780	0.304P	7.923	3.090P
				Σ10.251P	Σ20.502	Σ31.996P	Σ32.000	Σ41.004P

$$\Delta_x=\sum\frac{Mm_h ds}{EI}=+31.996P\times.000000299=.000009567P=.00000957P,$$

$$\Delta_y=\sum\frac{Mm_v ds}{EI}=\quad 41.004P\times.000000299=.000012260\;=.00001226P,$$

$$\Delta_a=\sum\frac{M ds}{EI}=\quad 10.251P\times.000000299=.000003065P=.00000306P.$$

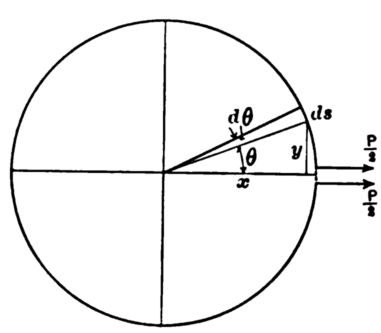


FIG. 29c.

Having computed these quantities by an approximate but general method, we will now compute them by the aid of the calculus:

$$\int\frac{Mm_h ds}{EI}=\int_0^\pi\frac{P}{2}\cdot\frac{r\sin\theta.1.r\sin\theta}{EI}.rd\theta=\frac{P}{2EI}\int_0^\pi r^3\sin^2\theta d\theta=\frac{Pr^3}{2EI}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^\pi=\frac{P\pi r^3}{4EI}$$

$$=.000009568P=.00000957P,$$

$$\int \frac{Mm_v ds}{EI} = \int_0^{\pi} \frac{P}{2} . r \sin \theta . 1 . r(1 - \cos \theta) r d\theta = \frac{Pr^3}{2EI}\left[\int_0^{\pi} \sin \theta d\theta - \int_0^{\pi} \cos \theta \sin \theta d\theta\right]$$

$$= \frac{Pr^3}{2EI}\left[-\cos \theta - \tfrac{1}{2} \sin^2 \theta\right]_0^{\pi} = \frac{Pr^3}{EI} = .000012190P = .00001219P,$$

$$\int \frac{Mds}{EI} = \int_0^{\pi} \frac{P}{2} . \frac{r \sin \theta}{EI} r . d\theta = -\frac{Pr^2}{2EI}\int_0^{\pi} - \cos \theta = +\frac{Pr^2}{EI}$$

$$= .000003048P = +.00000305P.$$

In computing stresses due to bending, the deflections due to certain unit quantities are also required. In the following are defined all such deflections for the solution of problems of curved bars subject to flexure. See Fig. 29*b* in connection with the following notation:

Let δ_{1v} = vertical deflection of *b* due to a vertical force of unity at *b*;
δ_{1h} = horizontal deflection of *b* due to a vertical force of unity at *b*;
δ_{1a} = angular motion of the plane *fn* due to a vertical force of unity at *b*;
δ_{2v} = vertical deflection of *b* due to a horizontal force of unity at *b*;
δ_{2h} = horizontal deflection of *b* due to a horizontal force of unity at *b*;
δ_{2a} = angular motion of the plane *fn* due to a horizontal force of unity at *b*;
δ_{3v} = vertical deflection of *b* due to moment of 1 in.-pound at *b*;
δ_{3h} = horizontal deflection of *b* due to moment of 1 in.-pound at *b*;
δ_{3a} = angular motion of the plane *fn* due to moment of 1 in.-pound at *b*.

The effect of a moment of one inch-pound in producing the deflections δ_{3v}, δ_{3h}, and δ_{3a} may be more clearly understood by a study of Fig. 29*d*, in which

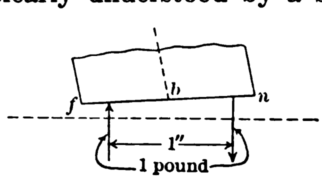


FIG. 29*d*.

the moment of one inch-pound is represented by a couple each force of which is one pound and the lever arm one inch. Throughout this book a moment will be represented by a couple.

A convenient tabulation of the computation for the above quantities follows:

It should be noted that the angular deflections are expressed as the length of an arc at the extremity of a unit radius and are measured in the same unit as the radius.

TABLE No. 29b

Div. No.	$\delta_{1v}=\frac{m_v^2 ds}{EI}$	$\delta_{1h}=\delta_{2v}=\frac{m_v m_h ds}{EI}$	$\delta_{2h}=\frac{m_h^2 ds}{EI}$	$\delta_{3v}=\delta_{1a}=\frac{m_v ds}{EI}$	$\delta_{3h}=\delta_{2a}=\frac{m_h ds}{EI}$	$\delta_{3a}=\frac{1 ds}{EI}$
1	$0.059\frac{ds}{EI}$	$0.060\frac{ds}{EI}$	$0.608\frac{ds}{EI}$	$0.077\frac{ds}{EI}$	$0.780\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
2	$0.454\frac{ds}{EI}$	$1.498\frac{ds}{EI}$	$4.936\frac{ds}{EI}$	$0.674\frac{ds}{EI}$	$2.222\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
3	$3.161\frac{ds}{EI}$	$5.914\frac{ds}{EI}$	$11.064\frac{ds}{EI}$	$1.778\frac{ds}{EI}$	$3.326\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
4	$10.368\frac{ds}{EI}$	$12.628\frac{ds}{EI}$	$15.388\frac{ds}{EI}$	$3.220\frac{ds}{EI}$	$3.923\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
5	$22.848\frac{ds}{EI}$	$18.756\frac{ds}{EI}$	$15.388\frac{ds}{EI}$	$4.780\frac{ds}{EI}$	$3.923\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
6	$38.713\frac{ds}{EI}$	$20.694\frac{ds}{EI}$	$11.064\frac{ds}{EI}$	$6.222\frac{ds}{EI}$	$3.326\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
7	$53.670\frac{ds}{EI}$	$16.278\frac{ds}{EI}$	$4.936\frac{ds}{EI}$	$7.326\frac{ds}{EI}$	$2.222\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
8	$62.774\frac{ds}{EI}$	$6.180\frac{ds}{EI}$	$0.608\frac{ds}{EI}$	$7.923\frac{ds}{EI}$	$0.780\frac{ds}{EI}$	$1.000\frac{ds}{EI}$
Σ 1 to 8 =	$192.047\frac{ds}{EI}$	$82.008\frac{ds}{EI}$	$63.992\frac{ds}{EI}$	$32.000\frac{ds}{EI}$	$20.502\frac{ds}{EI}$	$8.000\frac{ds}{EI}$
or =	0.000057422	0.00002452	0.000019134	0.000009568	0.000006130	0.000002392

Having computed these by the approximate method, they will be checked by the more exact method of the calculus:

$$\delta_{1v}=\int\frac{m_v^2 ds}{EI}=\int_0^\pi r^2\frac{(1-\cos\theta)^2}{EI}rd\theta=\int_0^\pi r^3\frac{(1-2\cos\theta+\cos^2\theta)}{EI}d\theta$$

$$=\frac{r^3}{EI}\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^\pi=\frac{3\pi r^3}{2EI}=.000057442,$$

$$\delta_{1h}=\int\frac{m_v m_h ds}{EI}=\frac{2r^3}{EI}=.000024383,$$

$$\delta_{2h}=\int\frac{m_h^2 ds}{EI}=\frac{\pi r^3}{2EI}=.000019136,$$

$$\delta_{3v}=\int\frac{m_v ds}{EI}=\int_0^\pi r\frac{(1-\cos\theta)}{EI}rd\theta=\frac{r^2}{EI}\left[\theta+\sin\theta\right]_0^\pi=\frac{\pi r^2}{EI}=.000009574,$$

$$\delta_{3h}=\int\frac{m_h ds}{EI}=-\frac{2r^2}{EI}=-.000006095,$$

$$\delta_{3a}=\frac{rd\theta}{EI}=\frac{\pi r}{EI}=.000002394.$$

PROBLEM

No. **29a**. As a preliminary to Problem 30a compute all the corresponding deflections for the ring of Problem 30a which have just been computed in this article.

ART. 30. THE GENERAL EQUATIONS OF CONDITION FOR FINDING THE STRESSES IN A CIRCULAR RING UNDER TWO EQUAL AND OPPOSITE FORCES

It is evident that if we supply at the cut ends of the ring, of Art. 29, forces and a moment necessary to bring the cut ends of the ring together, then these are the stresses and moments at this cross-section in the ring when it is acting as a whole. No matter what the nature of the internal stresses at b they may be represented by a single force and a couple, or by the horizontal and vertical components of the force and the couple.

It is also clear that for the most general values of Δ_x, Δ_y, and Δ_a, that is, values determined for an unsymmetrical ring either with respect to load or shape or with respect to both load and shape, that the forces and moments necessary to bring the planes fn in contact will be H_b, V_b, and M_b, as indicated in Fig. 30a, of which either the forces or the moments may act as indicated, or in the

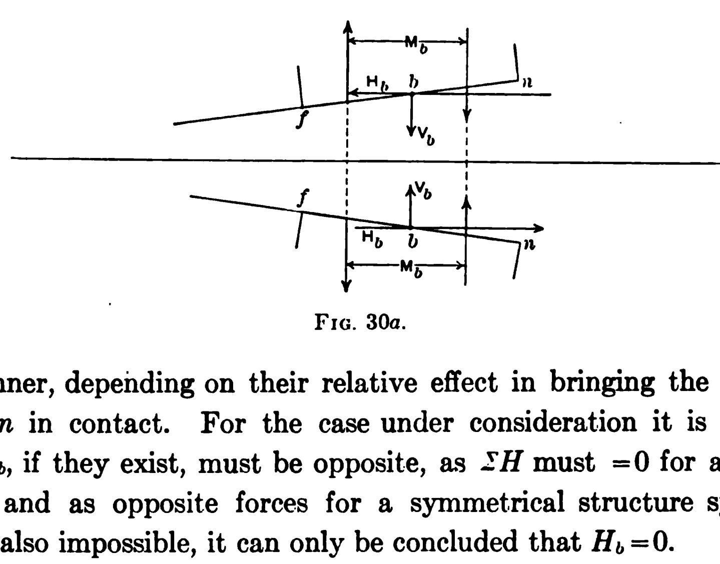


FIG. 30a.

opposite manner, depending on their relative effect in bringing the points b and the planes fn in contact. For the case under consideration it is clear that as the forces H_b, if they exist, must be opposite, as ΣH must $=0$ for a structure in equilibrium, and as opposite forces for a symmetrical structure symmetrically deformed, is also impossible, it can only be concluded that $H_b = 0$.

With the values of the deflections which have been previously defined and determined, equations will be written from which V_b and M_b may be found. The vertical and angular motion of the point b and the plane fn for this problem must respectively $=0$.

$$\therefore \quad \Delta_y - V_b\delta_{1v} - M_b\delta_{3v} = 0, \quad \ldots\ldots\ldots \quad (1)$$

and

$$\Delta_a - V_b\delta_{1a} - M_b\delta_{3a} = 0. \quad \ldots\ldots\ldots \quad (2)$$

From which equations both V_b and M_b may be found in character and amount.

It may be clearer to write the equations for finding H_b, V_b, and M_b before showing that H_b must $=0$.

The final vertical motions of the points b of the upper and lower surfaces must be such as to bring these points together, as they are the same point for the uncut ring. The same may be said for the horizontal motions of b. The angular motions of the planes fn must also bring the planes together. These three facts, established by the nature of the problem, enable the three following equations to be written:

The downward motion of b of the upper surface

$$= -\Delta_y + V_b\delta_{1v} + H_b\delta_{2v} + M_b\delta_{3v},$$

and this equals the downward motion of b of the lower surface

$$= +\Delta_y - V_b\delta_{1v} + H_b\delta_{2v} - M_b\delta_{3v},$$

or

$$\Delta_y - V_b\delta_{1v} - M_b\delta_{3v} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The motion to the right of b of the upper surface

$$= \Delta_x - V_b\delta_{1h} - H_b\delta_{2h} - M_b\delta_{3h},$$

and this equals the motion to the right of b of the lower surface

$$= \Delta_x - V_b\delta_{1h} + H_b\delta_{2h} - M_b\delta_{3h},$$

or

$$-H_b\delta_{2h} = +H_b\delta_{2h},$$

which can only be possible when $H_b = 0$.

The clockwise motion of the upper plane fn

$$= -\Delta_a + V_b\delta_{1a} + H_b\delta_{2a} + M_b\delta_{3a},$$

and this equals the clockwise motion of the lower plane fn

$$= +\Delta_a - V_b\delta_{1a} + H_b\delta_{2a} - M_b\delta_{3a},$$

or

$$\Delta_a - V_b\delta_{1a} - M_b\delta_{3a} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From either (1) and (2) or (3) and (4) we may write

$$V_b + M_b\frac{\delta_{3v}}{\delta_{1v}} = \frac{\Delta_y}{\delta_{1v}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$V_b + M_b\frac{\delta_{3a}}{\delta_{1v}} = \frac{\Delta_a}{\delta_{1a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Subtracting (6) from (5) we have

$$M_b\left[\frac{\delta_3}{\delta_{1v}}-\frac{\delta_{3a}}{\delta_{1a}}\right]=\frac{\Delta_v}{\delta_{1v}}-\frac{\Delta_a}{\delta_{1a}},$$

and

$$M_b=\frac{\Delta_v\delta_{1a}-\Delta_a\delta_{1v}}{\delta_{3v}\delta_{1a}-\delta_{1v}\delta_{3a}}. \qquad (a)$$

From (1) and (2) or (3) and (4) also

$$M_b+V_b\frac{\delta_{1v}}{\delta_{3v}}=\frac{\Delta_v}{\delta_{3v}}, \qquad (7)$$

and

$$M_b+V_b\frac{\delta_{1a}}{\delta_{3a}}=\frac{\Delta_a}{\delta_{3a}}. \qquad (8)$$

Subtracting (8) from (7),

$$V_b\left(\frac{\delta_{1v}}{\delta_{3v}}-\frac{\delta_{1a}}{\delta_{3a}}\right)=\frac{\Delta_v}{\delta_{3v}}-\frac{\Delta_a}{\delta_{3a}}=\frac{\Delta_v\delta_{3a}-\Delta_a\delta_{3v}}{\delta_{3v}\delta_{3a}},$$

and

$$V_b=\frac{\Delta_v\delta_{3a}-\Delta_a\delta_{3v}}{\delta_{1v}\delta_{3a}-\delta_{3v}\delta_{1a}}. \qquad (b)$$

Introducing in (a) and (b) the numerical values for the various terms we have

$$M_b=\frac{(.00001226\times.00000957-.00000306\times.00005742)P}{(.00000957\times.00000957-.\ 0005742\times.00000239)}=\frac{-58P}{-45}=1.29P,$$

and

$$V_b=\frac{(.00001226\times.00000239-.00000306\times.00000957)P}{(.00005742\times.00000239-.00000957\times.00000957)}=\frac{0P}{45}=0.$$

Substituting in (a) and (b) the general values derived for the calculus, we have

$$M_b=\frac{\dfrac{Pr^3}{EI}\cdot\dfrac{\pi r^2}{EI}-\dfrac{Pr^2}{EI}\cdot\dfrac{3\pi r^3}{2EI}}{\dfrac{\pi r^2}{EI}\cdot\dfrac{\pi r^2}{EI}-\dfrac{3\pi r^3}{2EI}\cdot\dfrac{\pi r}{EI}}=\frac{\dfrac{P\pi r^5}{(EI)^2}-\dfrac{3P\pi r^5}{2(EI)^2}}{\dfrac{\pi^2r^4}{(EI)^2}-\dfrac{3\pi^2r^4}{2(EI)^2}}=\frac{P\pi r^5(2-3)}{\pi^2r^4(2-3)}=\frac{Pr}{\pi},$$

and

$$V_b=\frac{\dfrac{Pr^3}{EI}\cdot\dfrac{\pi r}{EI}-\dfrac{Pr^2}{EI}\cdot\dfrac{\pi r^2}{EI}}{\dfrac{3}{2}\dfrac{\pi r^3}{EI}\dfrac{\pi r}{EI}-\dfrac{\pi r^2}{EI}\cdot\dfrac{\pi r^2}{EI}}=\frac{\dfrac{P\pi r^4}{(EI)^2}-\dfrac{P\pi r^4}{(EI)^2}}{\dfrac{3}{2}\dfrac{\pi^2r^4}{(EI)^2}-\dfrac{\pi^2r^4}{(EI)^2}}=0.$$

The calculations having been carried through by means of the approximate methods of integrating and the values thus determined used to find the ring stresses and moments, to familiarize the student with these approximate methods, which are entirely general and are the only methods generally applicable to the

problems of practice, in which problems the axis of the curved bar is generally not a specific curve, and the moment of inertia of the cross-section varies throughout the length of the axis. The ring selected was taken as circular and of uniform cross-section and as this gives us differential equations which are readily integrable, we can readily compare the results by the two methods and see the error due to the approximate integration.

$V_b = 0$ by both the exact and approximate methods;

$M_b = 1.29P$ by the approximate method; and

$= \dfrac{4P}{3.14} = 1.28P$ by the more exact method.

With these values for V_b and M_b it is a very simple matter to investigate the moments at any other point.

The moment at point a of Fig. 30b,

$$= M_b - \frac{P}{2} . r \sin\theta = \frac{Pr}{\pi} - \frac{P}{2} r \sin\theta .$$

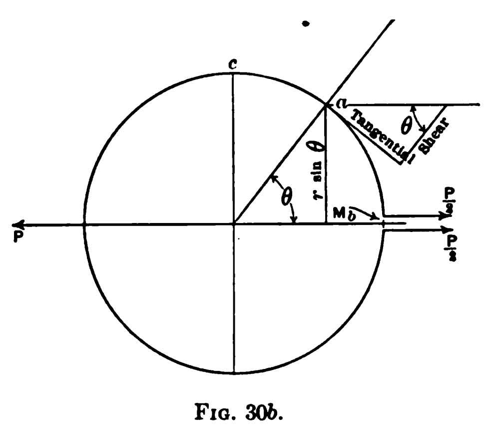


FIG. 30b.

This gives a maximum positive value at b of $\dfrac{Pr}{\pi}$ and a maximum negative value at c of $-.18Pr$. The maximum unit stress at any point is that due to bending plus that due to the tangential component of $\dfrac{P}{2}$. The tangential component $= \dfrac{P}{2} \sin\theta$.

The stress per square inch at c

$$= \frac{P}{2} \div A + \frac{Mc}{I} = .34P + .72 \times 3.92P = .34P + 2.82P = 3.16P.$$

The stress per square inch at b

$$=\frac{Mc}{I}=1.28P\times 3.92=5.02P.$$

PROBLEMS

No. 30*a*. Compute the maximum unit stresses in the ring of chain which is lifting 20,000 lbs., the ring being made of $1\frac{1}{2}$ round wrought iron and to a diameter of ring of 6 ins. Use the approximate method and check by means of the calculus.

No. 30*b*. If V_b, H_b, and M_b act as shown in Fig. 30*a* and if $V_b=10{,}000$ lbs., $H_b=1000$ lbs. and $M_b=10{,}000$ inch-pounds, find the amount and line of action of the resultant.

ART. 31. THE DESIGN OF A PIPE CULVERT

The method which has been used to find the stresses in the circular ring of the previous article may readily be applied to finding the stresses in a great variety of structures for the problems of practice. For example, in sewers, tunnel linings, and pipe culverts. The real difficulty in the way of the solution of the problem of design of such structures is that the nature and amount of the external forces are

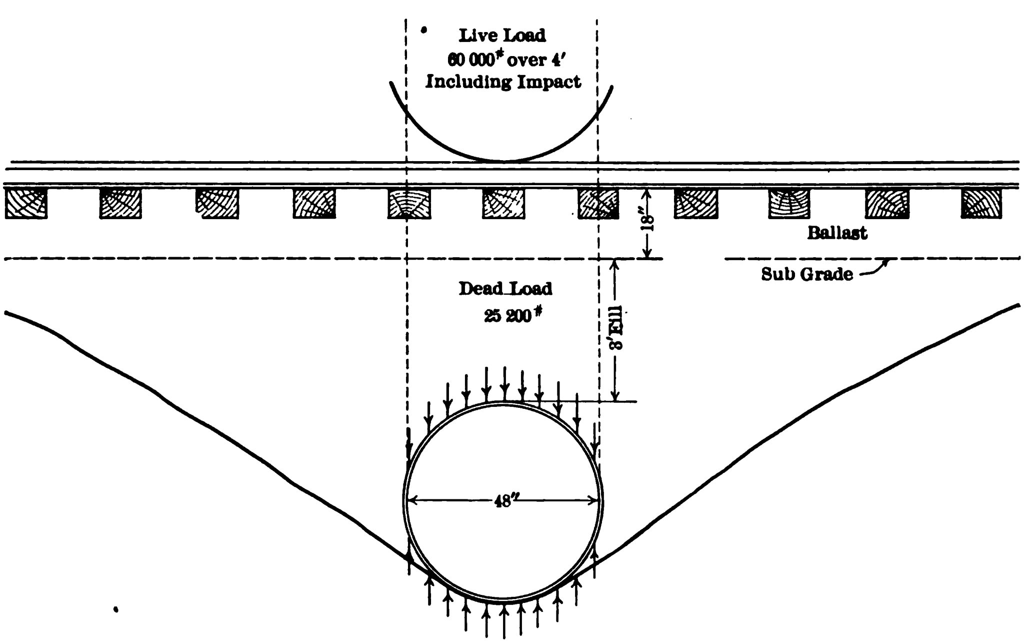


Fig. 31*a*.

indeterminate rather than the internal stresses. Fig. 31*a* shows a pipe culvert passing through an embankment which carries a railway track. Assuming that the loading or outer forces may be represented by two sets of vertical forces each

of which is uniform over the horizontal projection of the diameter as is indicated in the figure, the method of the previous articles may be used to find the stresses in the pipe as will be illustrated in what follows in this article.

It should be noticed here that vertical forces such as are shown acting on the pipe of Fig. 31a would tend to produce an increase in the horizontal diameter of the pipe and in all horizontal dimensions and thereby cause the earth surrounding the pipe to resist such change, or in other words the earth would supply what is sometimes designated as passive pressure, in a horizontal direction to the pipe.

The assumed loading of Fig. 31a is probably more severe than the actual and the solution will be made in accordance with this assumption. To permit the use of the equations of Art. 30 the pipe and forces will be rotated through an angle of 90°, and to better show the resulting deformations when the pipe is considered cut at one point, the forces will be taken as acting away from the vertical diameter, all as shown in Fig. 31b.

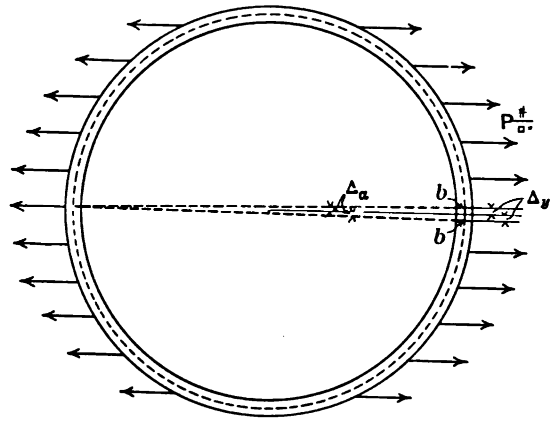


FIG. 31b.

In addition to the nomenclature of Arts. 29 and 30,

Let t = thickness of the shell of the pipe in inches;

r = radius of the axis of the shell of the pipe in inches;

P = pressure in lbs. per sq.in. on the horizontal projection of the shell axis;

$$M_b = \frac{\Delta_v \delta_{1a} - \Delta_a \delta_{1v}}{\delta_{3v}\delta_{1a} - \delta_{1v}\delta_{3a}}, \quad . \; . \; . \; . \quad (a) \text{ from Art. 30}$$

$$V_b = \frac{\Delta_v \delta_{3a} - \Delta_a \delta_{3v}}{\delta_{1v}\delta_{3a} - \delta_{3v}\delta_{1a}}, \quad . \; . \; . \; . \quad (b) \text{ from Art. 30}$$

$$\left.\begin{aligned}\delta_{1v} &= \frac{3\pi r^3}{2EI} \\ \delta_{1a} = \delta_{3v} &= \frac{\pi r^2}{EI} \\ \delta_{3a} &= \frac{\pi r}{EI}\end{aligned}\right\} \text{from Art. 29}$$

$$\Delta_v = \int_0^\pi \frac{Mm_v ds}{EI} \qquad \Delta_a = \int_0^\pi \frac{Mds}{EI}$$

From Fig. 31c

$$M = Pr\sin\theta . \frac{r\sin\theta}{2} = \frac{Pr^2\sin^2\theta}{2}$$

$$m_v = 1.r(1-\cos\theta)$$

$$ds = rd\theta.$$

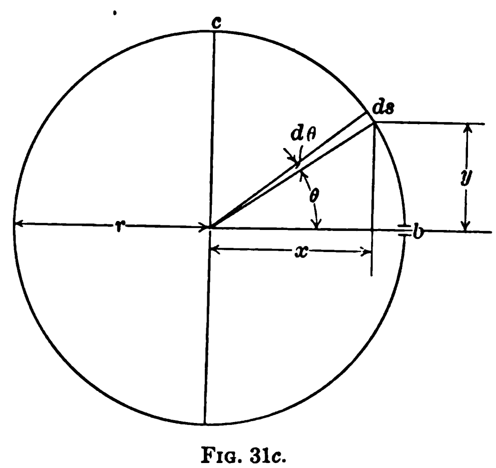


FIG. 31c.

Substituting these values of M, m_v, and ds and integrating

$$\Delta_v = \int_0^\pi \frac{Pr^2\sin^2\theta .1.r(1-\cos\theta)rd\theta}{2EI},$$

$$\Delta_v = \frac{Pr^4}{2EI}\int_0^\pi \sin^2\theta(1-\cos\theta)d\theta,$$

$$\Delta_v = \frac{Pr^4}{2EI}\left[\int_0^\pi \sin^2\theta d\theta - \int_0^\pi \sin^2\theta\cos\theta d\theta\right]$$

$$\int_0^\pi \sin^2\theta d\theta = \int_0^\pi\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi = \frac{\pi}{2}$$

$$\int_0^\pi \sin^2\theta\cos\theta d\theta = \int_0^\pi (1-\cos^2\theta)\cos\theta = \int_0^\pi \cos\theta d\theta - \int_0^\pi \cos^3\theta d\theta,$$

$$\int_0^\pi \cos\theta d\theta = \Big[\sin\theta\Big]_0^\pi = 0,$$

$$\int_0^\pi \cos^3\theta d\theta = \left[\frac{\sin\theta\cos^2\theta}{3}\right]_0^\pi + \frac{2}{3}\int_0^\pi \cos\theta d\theta = \left[\frac{\sin\theta\cos^2\theta}{3} + \frac{2}{3}\sin\theta\right]_0^\pi = 0,$$

$$\therefore \Delta_v = \frac{Pr^4}{2EI}\cdot\frac{\pi}{2} = \frac{Pr^4\pi}{4EI}$$

$$\Delta_a = \int_0^\pi \frac{Pr^2\sin^2\theta r d\theta}{2EI} = \frac{Pr^3}{2EI}\int_0^\pi \sin^2\theta d\theta,$$

$$\Delta_a = \frac{Pr^3}{2EI}\cdot\frac{\pi}{2} = \frac{Pr^3\pi}{4EI}.$$

Substituting in formulas (a) and (b) the values of the various deflections.

$$M_b = \frac{\dfrac{Pr^4\pi}{4EI}\dfrac{\pi r^2}{EI} - \dfrac{Pr^3\pi}{4EI}\dfrac{3\pi r^3}{2EI}}{\dfrac{\pi r^2}{EI}\dfrac{\pi r^2}{EI} - \dfrac{3\pi r^3}{2EI}\dfrac{\pi r}{EI}} = \frac{\dfrac{Pr^6}{4} - \dfrac{3Pr^6}{8}}{\dfrac{r^4}{1} - \dfrac{3r^4}{2}} = \frac{-\dfrac{1}{8}Pr^6}{-\dfrac{1}{2}r^4},$$

$$M_b = +\frac{Pr^2}{4},$$

$$V_b = \frac{\dfrac{Pr^4\pi}{4EI}\dfrac{\pi r}{EI} - \dfrac{Pr^3\pi}{4EI}\dfrac{\pi r^2}{EI}}{\dfrac{3\pi r^3}{2EI}\dfrac{\pi r}{EI} - \dfrac{\pi r^2}{EI}\dfrac{\pi r^2}{EI}} = 0.$$

M at any point on the axis $= M_b - \dfrac{Pr^2\sin^2\theta}{2} = \dfrac{Pr^2}{4} - \dfrac{Pr^2\sin^2\theta}{2}$,

$$M = \frac{Pr^2}{4}[1 - 2\sin^2\theta].$$

Maximum positive value of M occurs when $\theta = 0°$ or $180°$, or

$$M = \frac{Pr^2}{4} \text{ at } b,$$

Maximum negative value of M occurs when $\theta = 90°$ or $270°$, or

$$M = \frac{Pr^2}{4}[1-2] = -\frac{Pr^2}{4} \text{ at } c.$$

Thickness of Pipe Required for Maximum Stress at (b). At b there is only a stress due to the moment. Consider a section of pipe 1 in. long and of a thickness of t inches. The flexure formula is

$$S=\frac{Mc}{I},$$

where S = allowed unit stress in tension or compression $M=M_b$ and.

$$\frac{c}{I}=\frac{t12}{2t^3}=\frac{6}{t^2},$$

$$\therefore\quad S=\frac{6M_b}{t^2}=\frac{6Pr^2}{4t^2},\text{ and}$$

$$t^2=\frac{3Pr^2}{2S},\text{ and}$$

$$t=r\sqrt{\frac{3P}{2S}}.$$

Thickness of Pipe for Maximum Stress at (c). At c there is the negative moment and the direct stress.

$$\text{Negative moment}=-\frac{Pr^2}{4}\text{ in.lbs.}$$

$$\text{Direct Stress}\quad =P.1.r=Pr\text{ pounds.}$$

$$\therefore\quad \text{Max. Stress}=S=\frac{Pr}{A}+\frac{Mc}{I}\text{ where }A=\text{area}=1''\times t'',$$

$$S=\frac{Pr}{t}+\frac{Pr^2 6}{4t^2},$$

$$St^2=Prt+\frac{3Pr^2}{2},$$

$$St^2-Prt-\frac{3Pr^2}{2}=0.$$

Solving for t

$$t=\frac{+Pr\pm\sqrt{P^2r^2+4S\frac{3Pr^2}{2}}}{2S};$$

$$t=\frac{Pr\pm r\sqrt{P^2+6PS}}{2S}.$$

To find t required for case shown in Fig. 31a when the shell of the pipe is of cast iron:

$$r = 24 \text{ ins.}$$

$$S = 3000 \text{ lbs. per square inch in tension.}$$

$$S_1 = 10{,}000 \text{ lbs. per square inch in compression.}$$

$$P = \frac{85{,}200 \text{ lbs.}}{4' \times 13' \times 144} = 11.38 \text{ lbs. per square inch.}$$

At b

$$t = r\sqrt{\frac{3P}{2S}} = 24\sqrt{\frac{3 \times 11.38}{2 \times 3000}} = 24\sqrt{.00569},$$

$$t = 24(.07543) = 1.8103 \text{ ins.} = 1\tfrac{13}{16} \text{ ins.}$$

At c we must consider the direct compression; $\therefore$ use $S_1 = 10{,}000$.

$$t = \frac{Pr \pm r\sqrt{P^2 + 6PS_1}}{2S_1}.$$

$$t = \frac{24(11.38) + 24\sqrt{129.5 + 6(11.38)10{,}000}}{2 \times 10{,}000},$$

$$t = \frac{136.56 + 12\sqrt{682929.5}}{10{,}000} = \frac{136.56 + 12(826.4)}{10{,}000},$$

$$t = \frac{10053.36}{10{,}000} = 1.0053 \text{ ins.} = 1 \text{ in.}$$

$\therefore$ The tension in the pipe at b determines the required thickness as $1\tfrac{13}{16}$ ins.

To find t if the shell is made of *steel*:

$$S = 16{,}000 \text{ sq.ins. compression or tension.}$$

It is readily seen that the equation for t at b will give the largest value since S is same for each case.

$$\therefore \quad t = r\sqrt{\frac{3P}{2S}},$$

$$t = 24\sqrt{\frac{3(11.38)}{2 \times 16{,}000}} = 24\sqrt{.001067},$$

$$t = 24(.03266) = 0.7838 \text{ ins.},$$

$$= \tfrac{13}{16} \text{ ins. to the nearest } \tfrac{1}{16} \text{ in.}$$

ART. 32. STRESSES IN THE STEEL FRAMING POR A TUNNEL LINING

The design of tunnel linings is such a very important problem that it will be well to devote a little space here to showing how the equations of Art. 30, or similar equations, may be applied to finding the stresses in such a structure. Let Fig. 32*a* show the framework supporting such a lining and Fig. 32*b* one set of vertical forces on the frame. These forces are the live and dead load above the roof

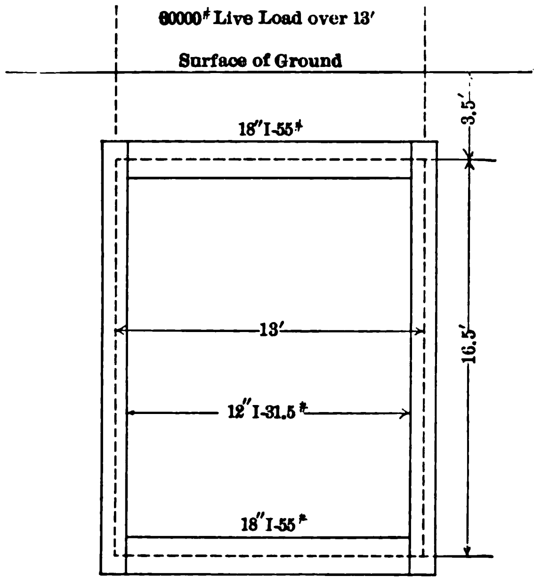


FIG. 32*a*.

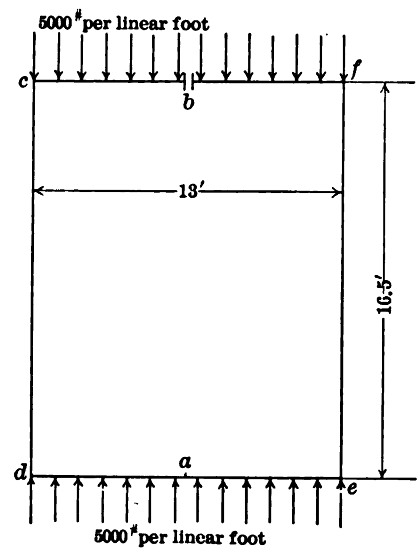


FIG. 32*b*.

and the reaction for the same below. Consider Fig. 32*b* to be revolved through 90 degrees in order that the equations of Art. 30 may be used without change. Whence:

$$\left.\begin{aligned} &\text{The Moment at } b = M_b = \frac{\Delta_v\delta_{1a} - \Delta_a\delta_{1v}}{\delta_{3v}\delta_{1a} - \delta_{1v}\delta_{3a}} \quad . \quad (a) \\ &\text{Vertical force at } b = V_b = \frac{\Delta_v\delta_{3a} - \Delta_a\delta_{3v}}{\delta_{1v}\delta_{3a} - \delta_{3v}\delta_{1a}} \quad . \quad (b) \\ &\text{Horizontal force at } b \quad = 0. \end{aligned}\right\} \quad \text{from Art. 30}$$

The shape of this structure is such that the effects of shear and direct stress in producing deflections may be readily included along with those due to bending.

The horizontal and vertical deflections at *b* with respect to *a* when the frame

is considered cut at b due to bending moment, shear and direct stress in the frame, are given in the following three general equations:

$$\Delta \text{ due to bending moment} = \int_b^a \frac{Mmds}{EI},$$

$$\Delta \text{ due to shear} = \int_b^a \frac{5Ssds}{2EA_w},$$

$$\Delta \text{ due to direct stress} = \int_b^a \frac{Ppds}{EA},$$

in which equations

S = shear on any section due to the given loads;

s = shear or any section due to unit load at b;

E_s for shear = $\frac{2}{5}E$ for direct stress;

P = direct stress due to the given loads;

p = direct stress due to the load of unity at b;

A = area of the cross-section of the beam;

A_w = area of web of I beams.

$$\Delta_v = \int_b^a \frac{Mm_v ds}{EI} + \int_b^a \frac{5Ss_v ds}{2EA_w} + \int_b^a \frac{Pp_v ds}{EA}$$

$$= \int_b^c \frac{\frac{5000x^2}{12(2)}.0.dx}{EI_1} + \frac{5}{2}\int_b^c \frac{\frac{5000}{12}x.0.dx}{EA_{1w}} + \int_b^c \frac{0.1.dx}{EA_1}$$

$$+ \int_c^d \frac{5000(6.5)(39)1.x.dx}{EI_2} + \frac{5}{2}\int_c^d \frac{0.1.dx}{EA_{2w}} + \int_c^d \frac{5000(6.5)0dx}{EA_2}$$

$$+ \int_d^a \frac{\frac{5000x^2}{12(2)}.1.(198)dx}{EI_1} + \frac{5}{2}\int_d^a \frac{5000x.0.dx}{12EA_{1w}} + \int_d^a \frac{0.1.dx}{EA_1},$$

$$\Delta_v = \int_c^d \frac{5000(6.5)(39)xdx}{EI_2} + \int_d^a \frac{5000(198)x^2dx}{12(2)EI_1}.$$

E = 30,000,000.

I_1 = for 18 ins. 55 lbs. – I = 795.6.

A_1 = 15.93 sq.ins.

A_{1w} = 8.28 sq.ins.

I_2 = for 12 ins. 31.5 lbs. – I = 215.8.

A_2 = 9.26 sq.ins.

A_{2w} = 4.20 sq.ins.

$$\Delta_v=\left[\frac{5000(6.5)39x^2}{2E\,215.8}\right]_{c=0}^{d=198}+\left[\frac{5000(198)x^3}{3(12)(2)E\,795.6}\right]_{d=0}^{a=78},$$

$$\Delta_v=\frac{115{,}130{,}000+8{,}202{,}000}{E}=\frac{123{,}332{,}000}{E}=4.1111 \text{ ins. down.}$$

$$\Delta_a=\int_b^a\frac{Mds}{EI}+\left(\begin{matrix}\Delta \text{ due to shear}\\ \text{reduces to } 0\end{matrix}\right)+\left(\begin{matrix}\Delta \text{ due to direct stress}\\ \text{reduces to } 0\end{matrix}\right)$$

$$\Delta_a=\int_b^c\frac{\frac{5000x^2dx}{12\,(2)}}{EI_1}+\int_c^d\frac{5000(6.5)39dx,}{EI_2}+\int_d^a\frac{\frac{5000x^2dx}{12(2)}}{EI_1}$$

$$\Delta_a=2\left[\frac{5000x^3}{3(12)2EI_1}\right]_0^{78}+\left[\frac{5000\,(6.5)39x}{EI_2}\right]_0^{198},$$

$$\Delta_a=\frac{2(5000)(78)^3}{3(12)2E\,795.6}+\frac{5000(6.5)39(198)}{E\,215.8},$$

$$\Delta_a=\frac{82{,}840+1{,}163{,}000}{E}=\frac{1{,}245{,}840}{E}=0.041528 \text{ clockwise.}$$

$$\delta_{1v}=\int_b^c\frac{0.0.dx}{EI_1}+\frac{5}{2}\int_b^c\frac{0.0.dx}{EsA_{1w}}+\int_b^c\frac{1.1.dx}{EA_1}$$
$$+\int_c^d\frac{1.x.1.xdx}{EI_2}+\frac{5}{2}\int_c^d\frac{1.1.dx}{E_sA_{2w}}+\int_c^d\frac{0.0.dx}{EA_2}$$
$$+\int_d^a\frac{1(198).1(198)dx}{EI_1}+\frac{5}{2}\int_d^a\frac{0.0.dx}{E_sA_{1w}}+\int_d^a\frac{1.1.dx}{EA_1},$$

$$\delta_{1v}=2\left[\frac{x}{EA_1}\right]_0^{78}+\left[\frac{x^3}{3EI_2}\right]_0^{198}+\left[\frac{5x}{2EA_{2w}}\right]_0^{198}+\left[\frac{(198)^2x}{EI_1}\right]_0^{78},$$

$$\delta_{1v}=\frac{1}{E}\left[\frac{2(78)}{15.93}+\frac{(198)^3}{3(215.8)}+\frac{5(198)}{2(4.2)}+\frac{(198)^2(78)}{795.6}\right],$$

$$\delta_{1v}=\frac{9.8+11990+235.7+3843.8}{E}=\frac{16079.3}{E},$$

$\delta_{1v}=0.00053598$ ins. downward.

For δ_{3v}, δ_{1a} and δ_{3a} the deflections due to shear and direct stress reduce to zero, $\therefore$ only the terms for bending are recorded in the following:

$$\delta_{1a}=\delta_{3v}=\int_b^c\frac{1.0.dx}{EI_1}+\int_c^d\frac{1.1.xdx}{EI_2}+\int_d^a\frac{1.1.(198)dx}{EI_1},$$
$$=\left[\frac{x_2}{2EI_2}\right]_0^{198}+\left[\frac{198x}{EI_1}\right]_0^{78}=\frac{1}{E}\left[\frac{(198)^2}{2(215.8)}+\frac{198(78)}{795.6}\right],$$

$$\delta_{1a}=\delta_{3v}=\frac{90.84+19.41}{E}=\frac{110.25}{E}=.000003675,$$

which for δ_{1a} is clockwise for a downward force of unity, and for δ_{3v} is downward for a clockwise moment of 1 in.-lb.

$$\delta_{3a}=\int_b^c \frac{1.dx}{EI_1}+\int_c^d \frac{1.dx}{EI_2}+\int_d^a \frac{1.dx}{EI_1}$$

$$=2\left[\frac{x}{EI_1}\right]_0^{78}+\left[\frac{x}{EI_2}\right]_0^{198}=\frac{1}{E}\left[\frac{2(78)}{795.6}+\frac{198}{215.8}\right],$$

$$\delta_{3a}=\frac{0.1961+0.9176}{E}=\frac{1.1137}{E}=0.00000003712,$$

which is clockwise for a clockwise moment.

Substituting in formulas (*a*) and (*b*) the values of the various deflections

$$M_b=\frac{\frac{123,332,000}{E}\cdot\frac{110.25}{E}-\frac{1,245,840}{E}\cdot\frac{16079.3}{E}}{\frac{16,079.3}{E}\cdot\frac{1.1137}{E}-\frac{110.25}{E}\cdot\frac{110.25}{E}}$$

$$=\frac{13,598,000,000-20,032,000,000}{17,905-12,158}=-\frac{6,434,000,000}{5,747}.$$

$$M_b=-1,119,400 \text{ in.-lbs}=-93,280 \text{ ft.-lbs.}$$

$$V_b=\frac{\frac{123,332,000}{E}\cdot\frac{1.1137}{E}-\frac{1,245,840}{E}\cdot\frac{110.25}{E}}{\delta_{1v}\delta_{3a}-\delta_{3v}\delta_{1a}}$$

$$=\frac{137,360,000-137,360,000}{E[\delta_{1v}\delta_{3a}-\delta_{3v}\delta_{1a}]}=0.$$

$$V_b=0.$$

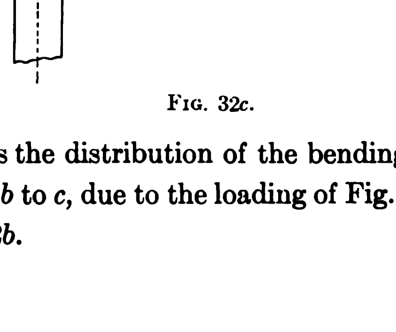


Fig. 32*c*.

Table No. 32*a* shows the distribution of the bending moments for various sections of the 18-in. I from *b* to *c*, due to the loading of Fig. 32*b*. Fig. 32*c* is a redrawing of a portion of Fig. 32*b*.

M at any point distant x from $b = \frac{Wx^2}{2} - M_b$;

$$W = 5000 \text{ lbs. per ft.}$$

$$M_b = 93{,}280 \text{ ft.-lbs.}$$

TABLE No. 32a

Values x in ft.	$\frac{Wx^2}{2}$ in ft.-lbs.	$\frac{Wx^2}{2} - M_b$ in ft.-lbs.	$\frac{Wx^2}{2} - M_b$ in in.-lbs.
1.5	5,625	−87,655	−1,051,860
2.5	15,625	−77,655	− 931,860
3.5	30,625	−62,655	− 751,860
4.5	50,625	−42,655	− 511,860
5.5	75,625	−17,655	− 211,860
6.5	105,625	+12,345	+ 148,140

In order to properly develop this and any other moment which may exist at c, a suitable connection should be provided at c between the top and sides of the lining frame.

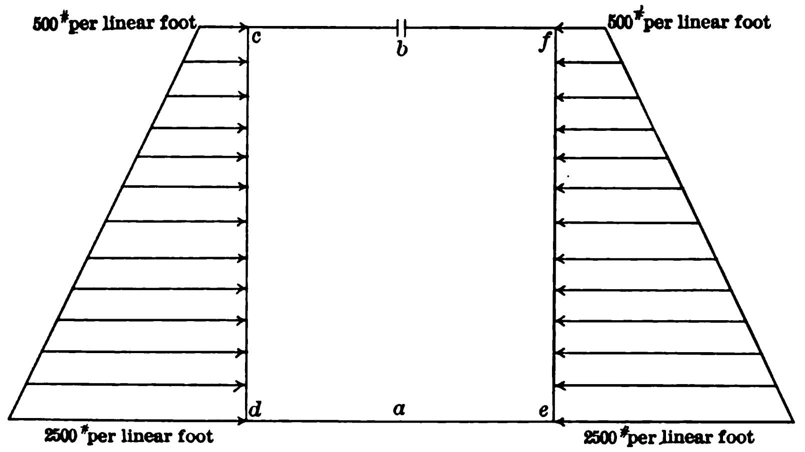


FIG. 32c.

PROBLEMS

No. **32a**. Compute the bending moment and direct stress at b of Fig. 32d when the structure is acting as a whole under the loading of this figure.

No. **32b**. Compute the bending moments in the tunnel lining of Fig. 32b by considering only the deformations due to bending and tabulate them as was done in Table 32a.

No. **32c**. Complete table 32a for the portion of the frame from c to d of Fig. 32b.

CHAPTER IV

ARCHES WITH SOLID WEBBED RIBS

ART. 33. THE DETERMINATION OF LIVE- AND DEAD-LOAD STRESSES IN THE ELASTIC ARCH WITH A SOLID WEB AND WITHOUT HINGES

THE structure without hinges has been selected as a first example because it is the form most used for solid webbed structures, and because it illustrates the method of making the computations in the most general form.

The computation of stresses in arches with solid webs cannot be made with the same degree of certainty as they can for either the open-webbed framework with articulate joints, or the straight beam. For the open framework, the stresses act in the lines joining the apices, and the modulus of elasticity for direct simple stress is fairly well known or may be readily determined. It is also possible to obtain the length of the various members with great exactness. It is therefore possible in the open framework to write equations of condition for finding stresses with a great degree of precision. The statement of the relation between stress and strain for all the elementary parts of the solid webbed arch is much more difficult, and in certain of its operations is not practically subject to great refinement. Let the sketch of Fig. 33*a*

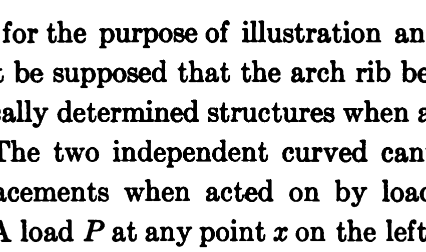


FIG. 33*a*.

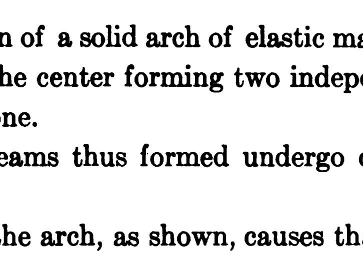


FIG. 33*b*.

show for the purpose of illustration an elevation of a solid arch of elastic material. Let it be supposed that the arch rib be cut at the center forming two independent statically determined structures when acting alone.

The two independent curved cantilever beams thus formed undergo certain displacements when acted on by loads.

A load P at any point x on the left half of the arch, as shown, causes that half of the arch to deflect and take a position with reference to its unloaded position as shown in Fig. 33*b*, where the full line figure represents the end of the unloaded

structure and the dotted line figure the loaded structure. It is clear that the stresses on the crown section when the arch acts as a whole are those necessary to bring the two cut ends (sections) together when the arch is considered to act as two independent cantilever curved beams.

No matter what the nature and amount of the stresses on the section of the arch at the crown when it is acting as a whole, they may be represented by a single force and a couple. As the single force may have any direction at the crown it is more convenient to consider it resolved into its horizontal and vertical components.

For the arch considered cut at the crown and with a load P at any point on the left half:

The stresses on any section of the left half are those due to the load P modified by the crown forces necessary to bring the cut ends together.

The stress on any section of the right half are those due to the crown forces necessary to bring the cut ends together;

In order to determine the unknown moment and forces necessary to bring the cut ends together the following nomenclature is defined:

Let P be any concentrated load at any point on the left half of the arch;
M be the bending moment at any point on the arch axis due to P;
m_v be the bending moment at any point on the arch axis due to a vertical load of unity at e;
m_h be the bending moment at any point on the arch axis due to a horital load of unity at e;
Δ_v be the vertical deflection of e due to P;
Δ_h be the horizontal deflection of e due to P;
Δ_a be the angular motion of the plane fn due to P;
δ_{1v} be the vertical deflection of e due to a vertical load of unity at e;
δ_{1h} be the horizontal deflection of e due to a vertical load of unity at e;
δ_{1a} be the angular motion of the plane fn due to a vertical load of unity at e;
δ_{2v} be the vertical deflection of e due to a horizontal force of unity at e;
$-\delta_{2h}$ be the horizontal deflection of e due to a horizontal force of unity at e;
$-\delta_{2a}$ be the angular motion of the plane fn due to a horizontal force of unity at e;
δ_{3v} be the vertical deflection of e due to a moment of 1 ft.-lb. at e;
$-\delta_{3h}$ be the horizontal deflection of e due to a moment of 1 ft.-lb. at e;
$-\delta_{3a}$ be the angular motion of the plane fn due to a moment of 1 ft.-lb. at e.

The resultant crown force will be completely determined when we know V_e its vertical component, H_e its horizontal component, and M_e the moment of H_e about

e. V_e, H_e and M_e will be taken as positive when they act as indicated in Fig. 33*c* and as negative when thay act in the opposite manner.

The downward motion of *e* for the left half $=\Delta_v - V_e\delta_{1v} - H_e\delta_{2v} - M_e\delta_{3v}$.

The downward motion of *e* for the right half $= +V_e\delta_{1v} - H_e\delta_{2v} - M_e\delta_{3v}$.

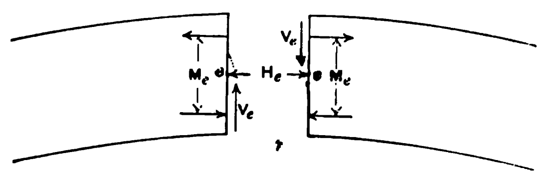


FIG. 33*c*.

These must be equal,

$$\therefore \quad 2V_e\delta_{1v} = \Delta_v. \qquad (1)$$

The motion to the right of *e* for the left half $=\Delta_h - V_e\delta_{1h} - H_e\delta_{2h} - M_e\delta_{3h}$.

The motion to the right of *e* for the right half $= -V_e\delta_{1h} + H_e\delta_{2h} + M_e\delta_{3h}$.

These must be equal,

$$\therefore \quad 2H_e\delta_{2h} + 2M_e\delta_{3h} = \Delta_h. \qquad (2)$$

The angular motion of the end plane *fn* clockwise for the left half

$$=\Delta_a - V_e\delta_{1a} - H_e\delta_{2a} - M_e\delta_{3a}.$$

The angular motion of *fn* clockwise for the right half

$$= -V_e\delta_{1a} + H_e\delta_{2a} + M_e\delta_{3a}.$$

These must be equal,

$$\therefore \quad 2H_e\delta_{2a} + 2M_e\delta_{3a} = \Delta_a. \qquad (3)$$

Solving equations (1), (2), (3) for V_e, H_e and M_e we have

$$V_e = \frac{\Delta_v}{2\delta_{1v}} = \Delta_v \times \frac{1}{2\delta_{1v}}, \qquad (4)$$

$$H_e = \frac{\Delta_h\delta_{3a} - \Delta_a\delta_{3h}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})} = \Delta_h \frac{\delta_{3a}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})} - \Delta_a \frac{\delta_{3h}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})}, \qquad (5)$$

$$M_e = \frac{\Delta_h\delta_{2a} - \Delta_a\delta_{2h}}{2(\delta_{3h}\delta_{2a} - \delta_{3a}\delta_{2h})} = -\Delta_h \frac{\delta_{2a}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})} + \Delta_a \frac{\delta_{2h}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})}, \qquad (6)$$

and these may be rewritten

$$V_e = \Delta_v a, \qquad (7)$$

$$H_e = \Delta_h b - \Delta_a c, \qquad (8)$$

and

$$M_e = -\Delta_h d + \Delta_a e, \qquad (9)$$

in which

$$a = \frac{1}{2\delta_{1v}},$$

$$b = \frac{\delta_{3a}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})},$$

$$c = \frac{\delta_{3h}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})},$$

$$d = \frac{\delta_{2a}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})},$$

and

$$e = \frac{\delta_{2h}}{2(\delta_{2h}\delta_{3a} - \delta_{2a}\delta_{3h})},$$

and in which a, b, c, d, and e are constants for any particular arch.

Later it will appear that $\delta_{2a} = \delta_{3h}$, and therefore $c = d$.

It should also be noted that the denominators for the values of b, c, d and e are identical.

ART. 34. TEMPERATURE STRESSES

The stresses in an arch with a solid web and without hinges due to temperature changes may be determined from consideration of the distortions the structure would undergo if divided into two statically determined structures, just as was done in determining the live- and dead-load stresses. The range of temperature through the entire mass of an arched rib of masonry is probably about $\pm 60°$ F. The possibility of having the live load and temperature acting together to produce maximum stress at any section is very remote. It therefore appears that ample security will be attained in a masonry arched bridge if a range of temperature of $\pm 50°$ F. be assumed.

For a rise in temperature:

Let Δ_{th} = the increase in length of the right or left half of the arch due to temperature increase;

Δ_{tv} = the increase in the rise of the right or left half of the arch due to temperature increase.

The angular rotation of the end plane, Δ_{at}, of either half due to a temperature increase $= 0$.

General equations similar to (1), (2), and (3), of Art. 33 may be written for finding V_{et}, H_{et}, and M_{et} due to temperature. Nomenclature not defined here is the same as for Art. 33.

The upward motion for the left half

$$=\Delta_{tv}+V_{et}\delta_{1v}+H_{et}\delta_{2v}+M_{et}\delta_{3v}.$$

The upward motion for the right half

$$=\Delta_{tv}-V_{et}\delta_{1v}+H_{et}\delta_{2v}+M_{et}\delta_{3v},$$

and these must be equal.

$$\therefore\quad 2V_{et}\times\delta_{1v}=0,\quad\text{and}\quad V_{et}=0.$$

The motion to the right of the left half

$$=\Delta_{th}-V_{et}\delta_{1h}-H_{et}\delta_{2h}-M_{et}\delta_{3h},$$

The motion to the right of the right half

$$=-\Delta_{th}-V_{et}\delta_{1h}+H_{et}\delta_{2h}+M_{et}\delta_{3h},$$

and these must be equal,

$$\therefore\quad H_{et}\delta_{2h}+M_{et}\delta_{3h}=\Delta_{th}. \qquad (10)$$

The clockwise motion of the end plane of the left half

$$=-V_{et}\delta_{1a}-H_{et}\delta_{2a}-M_{et}\delta_{3a},$$

The clockwise motion of the end plane of the right half

$$=-V_{et}\delta_{1a}+H_{et}\delta_{2a}+M_{et}\delta_{3a},$$

and these must be equal,

$$\therefore\quad H_{et}\delta_{2a}+M_{et}\delta_{3a}=0. \qquad (11)$$

From (10)

$$H_{et}\frac{\delta_{2h}}{\delta_{3h}}+M_{et}=\frac{\Delta_{th}}{\Delta_{3h}},$$

and from (11)

$$H_{et}\frac{\delta_{2a}}{\delta_{3a}}+M_{et}=0,$$

subtracting these

$$H_{et}\left(\frac{\delta_{2h}}{\delta_{3h}}-\frac{\delta_{2a}}{\delta_{3a}}\right)=\frac{\Delta_{th}}{\delta_{3h}},$$

and

$$H_{et}=\frac{\Delta_{th}}{\delta_{3h}\left[\frac{\delta_{2h}}{\delta_{3h}}-\frac{\delta_{2a}}{\delta_{3a}}\right]}=\Delta_{th}\frac{\delta_{3a}}{\delta_{2h}\delta_{3a}-\delta_{3h}\delta_{2a}}=H_{et}=2\Delta_{th}\times b. \qquad (12)$$

From (10)

$$H_{et}+M_{et}\frac{\delta_{3h}}{\delta_{2h}}=\frac{\Delta_{th}}{\delta_{2h}},$$

and from (11)

$$H_{et}+M_{et}\frac{\delta_{3a}}{\delta_{2a}}=0.$$

Subtracting these

$$M_{et}\left(\frac{\delta_{3h}}{\delta_{2h}}-\frac{\delta_{3a}}{\delta_{2a}}\right)=\frac{\Delta_{th}}{\delta_{2h}},$$

and

$$M_{et}=\frac{\Delta_{th}}{\delta_{2h}\left[\frac{\delta_{3h}}{\delta_{2h}}-\frac{\delta_{3a}}{\delta_{2a}}\right]}=\Delta_{th}\frac{\delta_{2a}}{\delta_{3h}\delta_{2a}-\delta_{3a}\delta_{2h}},$$

$$M_{et}=-2\Delta_{th}\times d. \qquad (13)$$

For a fall in temperature there may be derived in a similar manner

$$H_{et}=-2\Delta_{th}\times b, \qquad (14)$$

$$M_{et}=\ \ 2\Delta_{th}\times d. \qquad (15)$$

That is, the thrust and moment produced by a fall in temperature are equal and opposite to those due to a rise.

The coefficient of linear expansion for concrete is about .0000055 for each degree F.

$$\therefore \quad \Delta_{th}=50\times 50\times .0000055=.01375$$

for the arch assumed for illustrating Art. 37.

For a rise in temperature

$$H_{et}=2\times .01375\times 84{,}720=2330 \text{ lbs.},$$

and

$$M_{et}=-2\times .01375\times 510{,}200=-14{,}030 \text{ ft.-lbs.},$$

The eccentricity K of $H_{et}=\dfrac{14{,}030}{2330}=6.02$ ft.

ART. 35. STRESSES DUE TO RIB SHORTENING

The flexural stresses produce no change in the length of the neutral axis of the arch. The thrust, the resultant of all the forces on any given division, produces a shortening of the length of the arch axis. As the thrust (for each condition of live load and temperature) and cross-section of the rib vary at all points it would be a very long and laborious operation to include the effect of rib shortening in producing maximum stress on the arch cross-section at the center

of each division of the arch ring; for this reason its effect is generally only approximately included.

For any unsymmetrical application of the live load the thrust would make the arch axis unsymmetrical, and this would develop in general, horizontal and vertical forces, and a moment, at the crown.

For a symmetrical application of the live load the thrust does not deform the arch axis in an unsymmetrical manner, and the effect is similar to that of a fall in temperature.

The effect of rib shortening is not great in any case except for an arch with a small rise. It is generally sufficient to take the effect of the rib shortening for dead load and full live load, leaving rib shortening due to temperature out of consideration.

Let $S = \frac{8}{10}$ of the unit crown thrust due to full live and dead load. $\frac{8}{10}$ of this unit stress is used as the arch cross-section is smaller relatively to the thrust at the crown than at other points.

Δ_{sh} = shortening of half the span due to S;

H_{cs} = horizontal crown thrust due to S;

M_{cs} = moment at the crown due to S.

Then from the similarity to (14) and (15) we may write:

$$H_{cs} = -2\Delta_{sh} \times b \qquad (16)$$

$$M_{cs} = 2\Delta_{sh} \times d \qquad (17)$$

For the arch of the example used to illustrate Art. 37.

$$\Delta_{sh} = \frac{SL}{EA} = \frac{8}{10} \cdot \frac{(32{,}454+6886)}{324} \cdot \frac{50}{2{,}000{,}000} = \frac{39{,}340 \times 400}{324 \times 20{,}000{,}000} = .00243 \text{ ft.},$$

$$H_{cs} = -2 \times .00243 \times 84{,}720 = -412,$$

$$M_{cs} = 2 \times .00243 \times 510{,}200 = 2480,$$

$$K = \frac{2480}{412} = 6.02 \text{ ft.}$$

ART. 36. LIMITING POSITIONS FOR THE THRUST, FOR STRESSES OF THE SAME CHARACTER AS THE THRUST ON ANY CROSS-SECTION OF AN ARCH RIB

A method of determining the location and amount of the thrust, with reference to a particular cross-section, for a load at any point of the arch rib axis has been developed in the immediately preceding articles. With such a method developed and a knowledge of the stress distribution over a given cross-section when it is subject to direct or eccentric load, it will be a simple matter to select the points to be loaded to produce maximum stress at the extreme fibers of any cross-section. For example, if Fig. 36a be a portion of a rectangular rib of

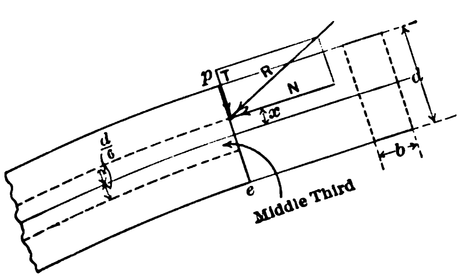


FIG. 36a.

homogeneous material the upper dotted line as drawn shows that every thrust above it produces tension at e, and the lower dotted line shows that every thrust below it produces tension at p, and further, the lower dotted line shows that every thrust above it produces compression at p and the upper dotted line shows that every thrust below it produces compression at e.

The dotted lines inscribed in the arch ring of Fig. 36a which determine whether the resultant of all the forces acting on any given section produces tensile or compressive stresses at the outer and inner fibers of the rib are correctly drawn only for a rib of rectangular cross-section and of homogeneous material, such as ribs of masonry, plain concrete, and cast iron. In order to make a somewhat complete presentation of the matter the manner of locating the dotted lines will be shown for three classes of ribs.

First. Ribs of homogeneous material and rectangular cross-section, which is here included only for completeness, as it is fully treated in many works on Applied Mechanics.

Second. Ribs of homogeneous material and I-beam cross-section.

Third. Ribs of two materials which under the ordinary conditions of

loading are so deformed that sections normal to the axis and plane before loading remain plane under the applied loads.

For the first case:

Let Fig. 36*a* show a portion of an arch rib.

Let *pe* be any plane section normal to the arch axis;

R be the resultant of all the forces acting on the section *pe*;

N and T be the normal and tangented components of R;

f be the unit stress on the extreme fiber of the cross-section;

A be the area of the cross-section of which *pe* is the trace $=bd$;

x be the lever arm of N with respect to the axis of the section.

Then

$$f=\frac{N}{bd}\pm\frac{Nx.6}{bd^2}=\frac{N}{bd}(1\pm\frac{6x}{d}),$$

in general, in which the first term represents the unit stress due to the direct action of N and the second that due to the moment of N. The stresses on the extreme fibers of any section on that side of the axis on which N acts are always increased and those on the opposite side of the axis decreased by the moment of N. Therefore, in order that all the stresses on the cross-section may have the same direction as N, the quantity in the parenthesis when taken with the minus sign for the second term must not pass through zero. The limiting value of x for stresses of the same direction as N on the extreme fibers of the cross-section on the opposite side of the axis to which N is applied is given by making

$$(1-\frac{6x}{d})=0, \quad \text{or when} \quad x=\frac{d}{6}.$$

For the second case, which is intended to represent that of a plate-girder rib of structural steel:

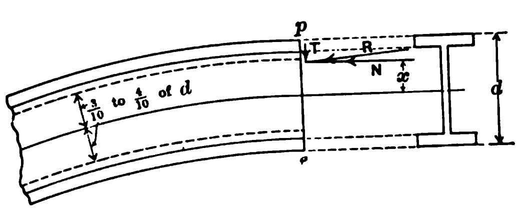


FIG. 36*b*.

Let Fig. 36*b* show a portion of such rib.

Let *pe*, R, N, T, f, and x be as defined for the first case;

A be the area of the cross-section;

I be the moment of inertia of the cross-section *pe*;

c be the distance of the extreme fiber of the cross-section from the neutral axis.

Then

$$f=\frac{N}{A}\pm Nx.\frac{c}{I}=N\left(\frac{1}{A}\pm\frac{x.c}{I}\right),$$

and the limiting value of x from the considerations stated for the first case is given from

$$\frac{1}{A}-\frac{x.c}{I}=0, \quad \text{or} \quad x=\frac{I}{A.c}.$$

x for a 24 in., 80 lb. I-beam of standard section $=\frac{2089.2}{23.32\times 12}=7.5$ ins. $=.31d$.

x for a 30 in., 200 lb. I-beam of Bethlehem section $=\frac{9154.7}{58.85\times 15}=10.4$ ins $=.35d$.

x for the center portion of the plate girder of Art. 24 $=\frac{100,526}{94\times 38.13}=30.5$ ins. $=.40d$.

From which it appears that x varies from $\frac{3}{10}$ to $\frac{4}{10}$ of d.

For the third case, which is intended to represent that of a rib of reinforced concrete, where the steel reinforcement is disposed symmetrically with reference to the axis of the rib. The design of this rib is also assumed to be made of such dimensions that there will be no cracks due to tension in the concrete on either the upper or lower surface of the rib.

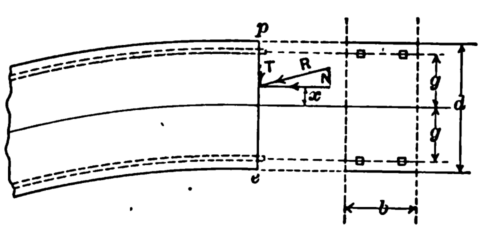


FIG. 36c.

Let Fig. 36c show a portion of such rib.

Let pe, R, N, T, c, and x be as defined for the first and second cases;

A = the area of a part of the rib between two vertical planes 12 ins. apart;

A_c = the area of concrete. (This may be taken as equal to A without appreciable error.)

A_s = the area of steel;

$p=\frac{A_s}{A_c}$ (1 to 2 per cent usually);

E_s = modulus of elasticity of steel;

E_c = modulus of elasticity of concrete;

$e=\frac{E_s}{E_c}$ (from 10 to 15 usually);

I = moment of inertia of the area A;

I_s = moment of inertia of the steel;

I_c = moment of inertia of the concrete;

f_c = the intensity of stresses on the concrete at the extreme fiber.

Then

$$f_c=\frac{N}{A}\pm\frac{N.x.c}{I}=\frac{N}{A_c+eA_s}\pm\frac{N.x.d}{2\left(\frac{bd^3}{12}+epbdg^2\right)}=\frac{N}{bd+e.p.bd}\pm\frac{6.N.x.d}{bd^3+12.e.pbdg^2}$$

$$=\frac{N}{bd}\left(\frac{1}{1+ep}\pm\frac{6.x.d}{d^2+12.e.pg^2}\right).$$

The limiting value of x from the considerations stated for the first case is given from

$$\frac{1}{1+ep}=\frac{6xd}{d^2+12.e.pg^2} \quad\text{or}\quad x=\frac{d^2+12.e.p.g^2}{6d(1+ep)}.$$

For a rib where $d=27$ ins., $b=12$ ins., $g=12$ ins., $e=10$ and $p=.01$,

$$x=\frac{729+12\times10\times\frac{1}{100}\times144}{6\times27\times1.1}=\frac{901.8}{178.2}=5.06 \text{ ins.} =.19d.$$

or a rib where $d=27$, $b=12$, $g=12$, $e=10$ and $p=.02$,

$$x=\frac{729+12\times10\times\frac{2}{100}\times144}{6\times27\times1.2}=\ldots=5.53 \text{ ins.} =.20d.$$

From which it appears that $x=\frac{d}{5}$ nearly, instead of $\frac{d}{6}$, as for the first case.

ART. 37. APPLICATION OF THE PREVIOUS METHOD TO DETERMINING THE STRESSES IN AN ARCH RING

In order to show the application of the previous method to arch analysis, the arch and loading of Figs. 37*a* and 37*b* are taken. The arch axis should be divided up into a suitable number of parts, and these parts made sufficiently small so that the work on each part may be obtained by taking the value of the moments and moment of inertia, at the center of the division, as constant throughout the division. Fig. 37*a* gives the dimensions assumed for the arch ring of Fig. 37*b*.

For an arch which receives its loads at definite points the divisions of the arch axis should be so made that the loaded points will be at or very near the centers of the divisions.

The best manner of dividing the arch axis into parts is to make a suitable number of divisions of the effective span length and by vertical lines through these division points divide the arch axis. The centers of the divisions of the axis will be a little farther away horizontally from the origin of coordinates,

when the origin is taken at the crown, than the centers of the divisions of the effective span length. This brings the point of application of the loads so near the centers of the divisions of the axis that they may be taken as applied at the centers of the divisions.

For the case assumed for illustration the arch span has been divided into twenty equal parts.

The values of the unit deflections and those for a load of $P=10{,}000$ lbs. at the center of each of the divisions of the half arch are first computed. The computations should be arranged in tabular form as shown on pages 92, 94 and 95. The values of V_e, H_e and M_e for a load of $P=10{,}000$ lbs. at the center of each of the divisions of the half arch should now be determined as is shown in Table No. 37*e*, then the equilibrium polygon for each of these loads is drawn as indicated in Fig. 37*c* of the insert facing page 98. For the problems of practice the equilibrium polygons should be drawn in an arch ring made to a scale of at least $\frac{1}{2}'' = 1'$. By means of this equilibrium polygon drawing, the loading giving maximum tensile and compressive stresses on any cross-section may be determined. With the condition of loading which produces maximum stresses at each cross-section known, the ten cross-sections may be investigated for strength.

For the purpose of illustration, let the cross-section at the center of Div. No. 4 be investigated.

The equilibrium polygon drawing of Fig. 37*c* shows that P_2 and all the loads to the left of it produce compression on the outer fiber and tension on the inner fiber at this section of the arch ring, because the thrusts for these loads lie above both the upper and lower edges of the middle third of the arch ring. The equilibrium drawing also shows that tension is produced on the outer fiber and compression on the inner fiber for P_1 and all loads to the right for the reason that the thrusts lie below both the upper and lower edges of the middle third of cross-section at this point.

The compression on the upper edge and tension on the lower edge of this section is increased by a fall in temperature and by rib shortening. The tension on the upper edge and compression on the lower edge is increased by a rise in temperature.

The conditions of loading for maximum compression on the upper and maximum tension on the lower fibers of Div. No. 4 are given in the sketch of Fig. 37*d*.

The conditions of loading for maximum compression on the lower and maximum tension on the upper fibers of Div. No. 4 are given in the sketch of Fig. 37*e*.

The computations for these maximum stresses are adjacent to the figures to which they apply and need no comment except to state that the values of the crown thrust and moment for Temp. and Rib Sh. are those computed in Arts. 34 and 35.

TABLE No. 37a

Ordinates of Centers of Sections with Reference to Axis at Crown.		
No.	X'	Y'
1	2.50	0.07
2	7.50	0.54
3	12.51	1.51
4	17.51	2.99
5	22.52	5.02
6	27.53	7.63
7	32.55	10.86
8	37.57	14.78
9	42.58	19.54
10	47.62	25.53

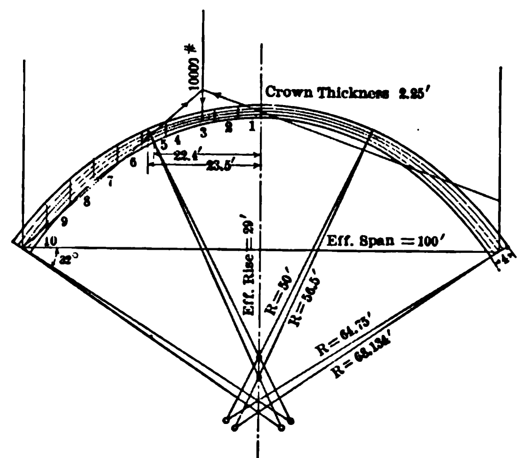


FIG. 37a

TABLE No. 37b

Div. No.	Depth of Ring in Feet.	ds in Feet.	I in Feet.[4]	$\frac{ds}{EI}$*	m_h for 1 lb.→ Ft.lbs.	m_v for 1 lb.↓ Ft.lbs.	M in Ft.lbs. for P Lbs. at									
							1	2	3	4	5	6	7	8	9	10
1	2.27	5.01	0.975	.1784	0.07	2.50	0	0	0	0	0	0	0	0	0	0
2	2.29	5.03	1.001	.1745	0.54	7.50	5.00P	0	0	0	0	0	0	0	0	0
3	2.38	5.16	1.123	.1595	1.51	12.51	10.01P	5.01P	0	0	0	0	0	0	0	0
4	2.50	5.30	1.302	.1413	2.99	17.51	15.01P	10.01P	5.00P	0	0	0	0	0	0	0
5	2.66	5.51	1.568	.1220	5.02	22.52	20.02P	15.02P	10.01P	5.01P	0	0	0	0	0	0
6	2.88	5.77	1.991	.1006	7.63	27.53	25.03P	20.03P	15.02P	10.02P	5.01P	0	0	0	0	0
7	3.09	6.15	2.459	.0868	10.86	32.55	30.05P	25.05P	20.04P	15.04P	10.03P	5.02P	0	0	0	0
8	3.33	6.60	3.077	.0745	14.78	37.57	35.07P	30.07P	25.06P	20.06P	15.05P	10.04P	5.02P	0	0	0
9	3.59	7.25	3.856	.0653	19.54	42.58	40.08P	35.08P	30.07P	25.07P	20.06P	15.05P	10.03P	5.01P	0	0
10	3.88	8.42	4.868	.0601	25.53	47.62	45.12P	40.12P	35.11P	30.11P	25.10P	20.09P	15.07P	10.05P	5.04P	0

* $\frac{ds}{EI}$ has been multiplied by 10,000,000 to avoid long decimals. $E = 2,000,000$ lbs. per square inch $= 288,000,000$ lbs. per square foot.

TABLE No. 37c

Div. No.	m_v for 1 lb.↓	m_v^2	$m_v^2\frac{ds}{EI}$	m_h for 1 lb.→	m_h^2	$m_h^2\frac{ds}{EI}$	$\frac{ds}{EI}$	$m_v m_h$	$m_v\frac{ds}{EI}$	$m_h\frac{ds}{EI}$	$m_v m_h\frac{ds}{EI}$
			δ_{1v}			δ_{2h}	δ_{3a}		$\delta_{1a}=\delta_{3v}$	$\delta_{2a}=\delta_{3h}$	$\delta_{1h}=\delta_{2v}$
1	2.50	6.25	1.115	0.07	0.0049	0.00087	0.1784	0.175	0.4460	0.0125	0.0312
2	7.50	56.25	9.816	0.54	0.2916	0.05089	0.1745	4.050	1.3087	0.0942	0.7067
3	12.51	156.50	24.962	1.51	2.2801	0.36370	0.1595	18.892	1.9954	0.2408	3.0130
4	17.51	306.60	43.325	2.99	8.9401	1.26320	0.1413	52.360	2.4743	0.4225	7.3980
5	22.52	507.15	61.895	5.02	25.2004	3.07430	0.1220	113.050	2.7477	0.6125	13.7920
6	27.53	757.90	76.250	7.63	58.2169	5.85700	0.1006	210.050	2.7698	0.7676	21.1300
7	32.55	1059.50	91.970	10.86	117.9300	10.23500	0.0868	353.500	2.8253	0.9426	30.6830
8	37.57	1411.50	105.150	14.78	218.4500	16.27300	0.0745	555.300	2.7988	1.1012	41.3720
9	42.58	1813.10	118.400	19.54	381.8000	24.92800	0.0653	832.000	2.7802	1.2758	54.3250
10	47.62	2267.70	136.280	25.53	651.8000	39.17000	0.0601	1215.600	2.8618	1.5343	73.0650
			669.163			101.21596	1.1630		23.0080	7.0040	245.5159

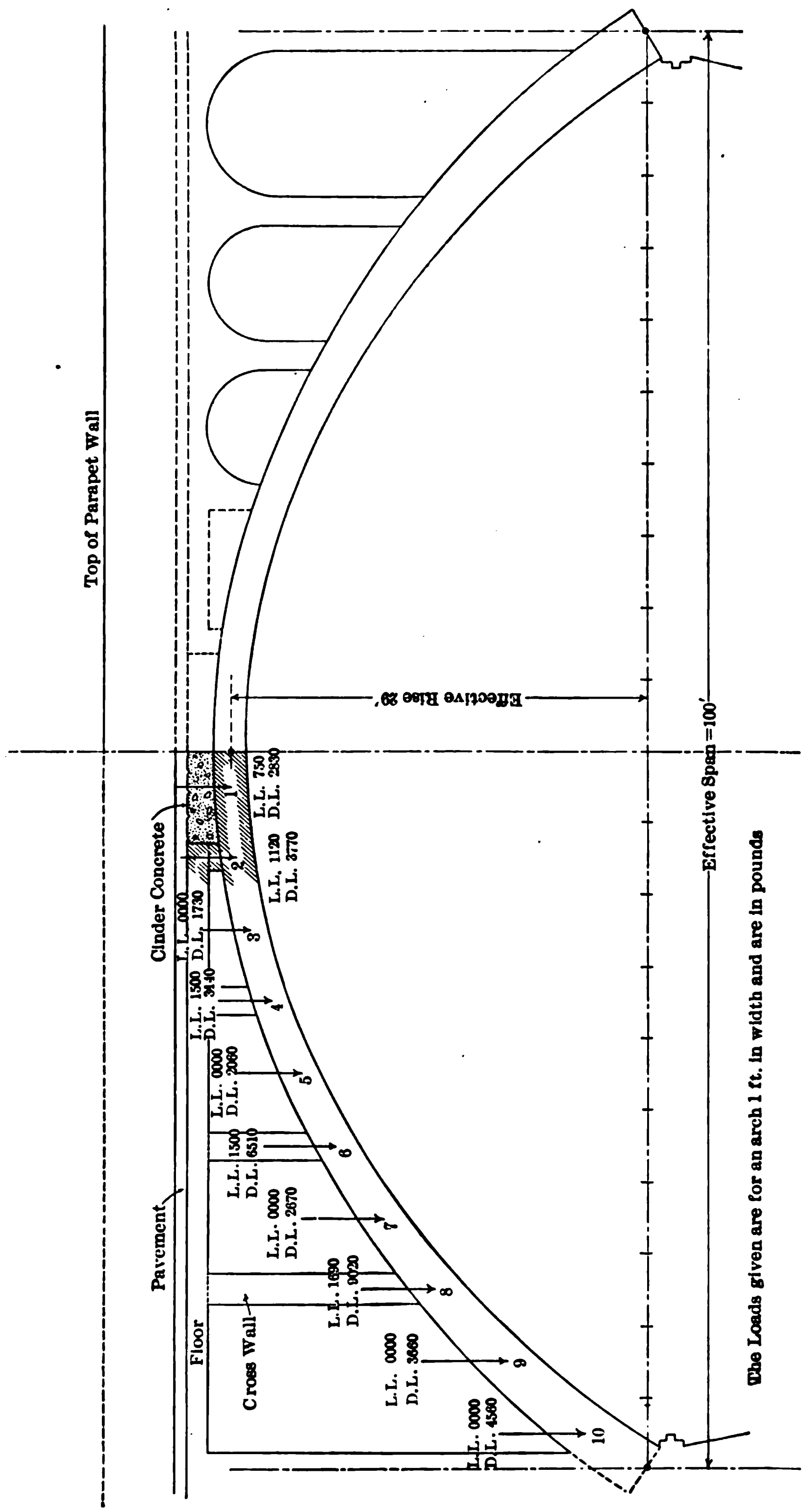

Top of Parapet Wall
Pavement
Floor
Cross Wall
Cinder Concrete
Effective Rise 29'
Effective Span=100'
L.L. 750 D.L. 2830
L.L. 1120 D.L. 3770
L.L. 0000 D.L. 1730
L.L. 1500 D.L. 3840
L.L. 0000 D.L. 2060
L.L. 1500 D.L. 6510
L.L. 0000 D.L. 2670
L.L. 1690 D.L. 9020
L.L. 0000 D.L. 3660
L.L. 0000 D.L. 4560
The Loads given are for an arch 1 ft. in width and are in pounds

Fig. 37b.

TABLE

Sect.	M	$\frac{ds}{EI}$	$M\frac{ds}{EI}$	m_v	m_h	$Mm_v\frac{ds}{EI}$	$Mm_h\frac{ds}{EI}$
	For P at 1		Δ_a			Δ_v	Δ_h
1	0	......	0	.....	.....	0	0
2	50,000	0.1745	8,725	7.50	0.54	65,440	4,710
3	100,100	0.1595	15,966	12.51	1.51	199,740	24,110
4	150,100	0.1413	21,209	17.51	2.99	371,400	63,420
5	200,200	0.1220	24,424	22.52	5.02	550,030	122,610
6	250,300	0.1006	25,180	27.53	7.63	693,250	192,120
7	300,500	0.0868	26,080	32.55	10.86	848,900	283,250
8	350,700	0.0745	26,127	37.57	14.78	981,600	386,150
9	400,800	0.0653	26,170	42.58	19.54	1,114,200	511,400
10	451,200	0.0601	27,115	47.62	25.53	1,291,200	692,200
			200,996			6,115,760	2,279,970
	For P at 3		Δ_a			Δ_v	Δ_h
1-3	0	......	0	......	......	0	0
4	50,000	0.1413	7,065	17.51	2.99	123,700	21,120
5	100,100	0.1220	12,212	22.52	5.02	275,000	61,300
6	150,200	0.1006	15,110	27.53	7.63	416,000	115,300
7	200,400	0.0868	17,395	32.55	10.86	566,200	188,900
8	250,600	0.0745	18,670	37.57	14.78	701,500	276,000
9	300,700	0.0653	19,630	45.58	19.54	835,900	383,600
10	351,100	0.0601	21,100	47.62	25.53	1,004,800	538,700
			111,182			3,923,100	1,584,920
	For P at 5		Δ_a			Δ_v	Δ_h
1-5	0	......	0	......	......	0	0
6	50,100	0.1006	5,040	27.53	7.63	138,740	38,450
7	100,300	0.0868	8,706	32.55	10.86	283,350	94,540
8	150,500	0.0745	11,212	37.57	14.78	421,200	165,720
9	200,600	0.0653	13,099	42.58	19.54	557,800	255,970
10	251,000	0.0601	15,085	47.62	25.53	718,400	385,140
			53,142			2,119,490	939,820
	For P at 7		Δ_a			Δ_v	Δ_h
1-7	0	......	0	......	......	0	0
8	50,200	0.0745	3,740	37.57	14.78	140,500	55,280
9	100,300	0.0653	6,550	42.58	19.54	278,880	127,980
10	150,300	0.0601	9,057	47.62	25.53	430,140	230,600
			19,347			849,520	413,860
	For P at 9		Δ_a			Δ_v	Δ_h
1-9	0	......	0	......	......	0	0
10	50,400	0.0601	3,029	47.62	25.53	144,241	77,330
			3,029			144,241	77,330

No. 37*d*

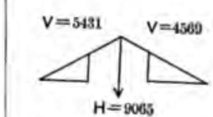
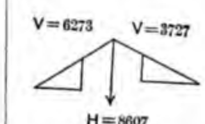
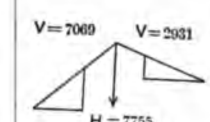
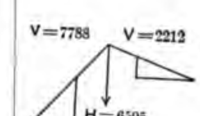
Sect.	M	$\frac{ds}{EI}$	$M\frac{ds}{EI}$	m_v	m_h	$Mm_v\frac{ds}{EI}$	$Mm_h\frac{ds}{EI}$
	For P at 2		Δ_a			Δ_v	Δ_h
1-2	0	......	0	......	......	0	0
3	50,100	0.1595	7,991	12.51	1.51	99,970	12,066
4	100,100	0.1413	14,144	17.51	2.99	247,680	42,295
5	150,200	0.1220	18,324	22.52	5.02	412,700	92,000
6	200,300	0.1006	20,150	27.53	7.63	554,750	153,300
7	250,500	0.0868	21,742	32.55	10.86	707,700	236,100
8	300,700	0.0745	22,402	37.57	14.78	841,700	331,130
9	350,800	0.0653	22,907	42.58	19.54	975,300	447,600
10	402,100	0.0601	24,112	47.62	25.53	1,148,200	615,600
			151,772			4,988,000	1,930,091
	For P at 4		Δ_a			Δ_v	Δ_h
1-4	0	......	0	......	......	0	0
5	50,100	0.1220	6,112	22.52	5.02	137,620	30,680
6	100,200	0.1006	10,080	27.53	7.63	277,500	76,910
7	150,400	0.0868	13,053	32.55	10.86	424,870	141,760
8	200,600	0.0745	14,945	37.57	14.78	561,500	220,900
9	250,700	0.0653	16,370	42.58	19.54	697,100	319,900
10	301,100	0.0601	18,096	47.62	25.53	861,800	462,000
			78,656			2,960,390	1,252,150
	For P at 6		Δ_a			Δ_v	Δ_h
1-6	0	......	0	......	......	0	0
7	50,200	0.0868	4,357	32.55	10.86	141,800	47,320
8	100,400	0.0745	7,480	37.57	14.78	281,000	110,550
9	150,500	0.0653	9,828	42.58	19.54	418,470	192,030
10	200,900	0.0601	12,074	47.62	25.53	575,000	308,250
			33,739			1,416,270	658,150
	For P at 8		Δ_a			Δ_v	Δ_h
1-8	0	......	0	......	......	0	0
9	50,100	0.0653	3,272	42.58	19.54	139,320	63,940
10	100,500	0.0601	6,040	47.62	25.53	287,580	154,180
			9,312			426,900	218,120
	For P at 10		Δ_a			Δ_v	Δ_h
1-10	0	......	0	......	......	0	0
			0			0	0

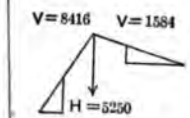
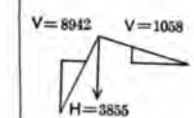
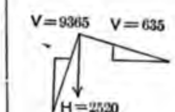
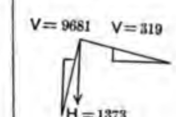
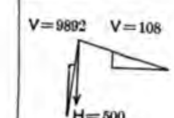
Table No. 37e

$$a=\frac{1}{2\delta_{1v}}=\frac{10{,}000{,}000}{2\times 669.163}=7472,\qquad d=\frac{\delta_{2a}}{2(\delta_{2h}\delta_{3a}-\delta_{2a}\delta_{3h})}=\frac{7.004\times 10{,}000{,}000}{2(101.216\times 1.163-[7.004]^2)}=+510{,}200,$$

$$b=\frac{\delta_{3a}}{2(\delta_{2h}\delta_{3a}-\delta_{2a}\delta_{3h})}=\frac{1.163\times 10{,}000{,}000}{2(101.216\times 1.163-[7.004]^2)}=+84{,}720,\qquad e=\frac{\delta_{2h}}{2(\delta_{2h}\delta_{3a}-\delta_{2a}\delta_{3h})}=\frac{101.216\times 10{,}000{,}000}{2(101.216\times 1.163-[7.004]^2)}=+7{,}372{,}000.$$

$$c=\frac{\delta_{3h}}{2(\delta_{2h}\delta_{3a}-\delta_{2a}\delta_{3h})}=\frac{7.004\times 10{,}000{,}000}{2(101.216\times 1.163-[7.004]^2)}=+510{,}200,$$

For load of 10000 lbs. at center of Division No.	1	$V_e=\Delta_v a=.611576\times 7472=4569$ $\quad K=\frac{31{,}850}{9{,}065}=3.514$ $H_e=\Delta_h b-\Delta_a c=.227997\times 84{,}720-.0200996\times 510{,}200=19{,}315-10{,}250=9065$ $M_e=\Delta_a e-\Delta_h d=.0200996\times 7{,}372{,}000-.227997\times 510{,}200=148{,}170-116{,}320=+31{,}850$	
do.	2	$V_e=\Delta_v a=.498800\times 7472=3727$ $\quad K=\frac{13{,}410}{8607}=1.558$ $H_e=\Delta_h b-\Delta_a c=.1930091\times 84{,}720-.0151772\times 510{,}200=16{,}351-7744=8607$ $M_e=\Delta_a e-\Delta_h d=.0151772\times 7{,}372{,}000-.1930091\times 510{,}200=111{,}880-98{,}470=+13{,}410$	
do.	3	$V_e=\Delta_v a=.392310\times 7472=2931$ $\quad K=\frac{1100}{7755}=0.142$ $H_e=\Delta_h b-\Delta_a c=.1584920\times 84{,}720-.0111182\times 510{,}200=13{,}428-5673=7755$ $M_e=\Delta_a e-\Delta_h d=.0111182\times 7{,}372{,}000-.158492\times 510{,}200=81{,}960-80{,}860=+1100$	
do.	4	$V_e=\Delta_v a=.296039\times 7472=2212$ $\quad K=\frac{-5900}{6595}=-0.895$ $H_e=\Delta_h b-\Delta_a c=.1252150\times 84{,}720-.0078656\times 510{,}200=10{,}608-4013=6595$ $M_e=\Delta_a e-\Delta_h d=.0078656\times 7{,}372{,}000-.125215\times 510{,}200=57{,}980-63{,}880=-5900$	

For load of 10000 lbs. at center of Division No.	5	$V_e=\Delta_v a=.211949\times 7472=1584$ $\quad K=\frac{-8775}{5250}=-1.671$ $H_e=\Delta_h b-\Delta_a c=.093982\times 84{,}720-.0053142\times 510{,}200=7962-2712=5250$ $M_e=\Delta_a e-\Delta_h d=.0053142\times 7{,}372{,}000-.093982\times 510{,}200=39{,}175-47{,}950=-8775$	
do.	6	$V_e=\Delta_v a=.141627\times 7472=1058$ $\quad K=\frac{-8710}{3855}=-2.259$ $H_e=\Delta_h b-\Delta_a c=.065815\times 84{,}720-.0033739\times 510{,}200=5576-1721=3855$ $M_e=\Delta_a e-\Delta_h d=.0033739\times 7{,}372{,}000-.065815\times 510{,}200=24{,}870-33{,}580=-8710$	
do.	7	$V_e=\Delta_v a=.084952\times 7472=635$ $\quad K=\frac{-6853}{2520}=-2.719$ $H_e=\Delta_h b-\Delta_a c=.041386\times 84{,}720-.0019323\times 510{,}200=3506-986=2520$ $M_e=\Delta_a e-\Delta_h d=.0019347\times 7{,}372{,}000-.041386\times 510{,}200=14{,}262-21{,}115=-6853$	
do.	8	$V_e=\Delta_v a=.042690\times 7472=319$ $\quad K=\frac{-4264}{1373}=-3.105$ $H_e=\Delta_h b-\Delta_a c=.021812\times 84{,}720-.0009312\times 510{,}200=1848-475=1373$ $M_e=\Delta_a e-\Delta_h d=.0009312\times 7{,}372{,}000-.021812-510{,}200=6864-11{,}128=-4264$	
do.	9	$V_e=\Delta_v a=.014424\times 7472=108$ $\quad K=\frac{-1712}{500}=-3.424$ $H_e=\Delta_h b-\Delta_a c=.007733\times 84{,}720-.0003029\times 510{,}200=655-155=500$ $M_e=\Delta_a e-\Delta_h d=.0003029\times 7{,}372{,}000-.007733\times 510{,}200=2233-3945=-1712$	
do.	10	$V_e=\Delta_v a=0\times 7472=0$ $\quad K=0$ $H_e=\Delta_h b-\Delta_a c=0\times 84{,}720-0\times 510{,}200=0$ $M_e=\Delta_a e-\Delta_h d=0\times 7{,}372{,}000-0\times 510{,}200=0$	

K = the distance from axis at the crown to the line of action of H_e.

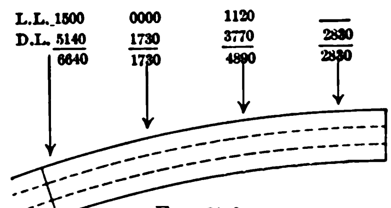


FIG. 37*d*.

V_4	H_4		M_4	
−6640	+35217	−412	+16827	−6518
−1730	−2742	−2330	+97100	−42478
−4890	——	——	+14030	−48949
−2830	+32475	−2742	+2480	−8650
−16090			+130437	−106595
+961			−106595	
−15129			+23842	

L.L. and D.L.

	V_e	H_e	M_e
D. L.	0000	+32,454	−5108
L. L.	+961	+2763	−1410
Total =	+961	+35,217	−6518

Fall in Temp.			Rib Shortening.		
V_{et}	H_{et}	M_{et}	V_{es}	H_{es}	M_{es}
0	−2330	+14,030	0	−412	+2480

Lever Arms; x's $=5.00'$, $10.01'$, $15.01'$ and $17.51'$, $y=2.99'$.

$$R_4=\sqrt{V_4^2+H_4^2}=\sqrt{15{,}129^2+32{,}475^2}=35{,}800.$$

Compression on upper fiber

$$=\frac{35{,}800}{2.5}+23{,}842\frac{1.25}{1.30}$$

$$=14{,}300+22{,}900=37{,}200 \text{ lbs. per sq.ft.}=260 \text{ lbs. per sq.in.}$$

Tension on lower fiber

$$=14{,}300-22{,}900=8600 \text{ lbs. per sq.ft.}=60 \text{ lbs. per sq.in.}$$

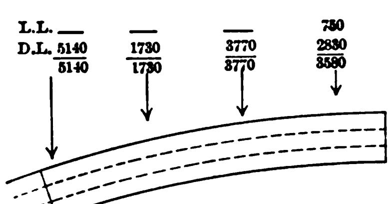


FIG. 37*e*.

V_4	H_4		M_4	
−961	+36577	−412	+2480	−1740
−3580	+2330		+115100	−14030
−3770	——		——	−16827
−1730	+38907		+117580	−53736
−5140	−412			−37738
				−8650
−15181	+38495			−132721
				+117580
				−15141

L.L. and D.L.

	V_e	H_e	M_e
D. L.	+0000	+32,454	−5108
L. L.	−961	+4123	+3368
Total =	−961	36,577	−1740

Rise in Temp.			Rib Shortening.		
V_{et}	H_{et}	M_{et}	V_{es}	H_{es}	M_{es}
0	+2330	−14,030	0	−412	+2480

$R_4=41{,}400.$

PLATE I.

ENCE TABLE FOR MAXIMUM COMPRESSION

Div. No. 4	Div. No. 5	Div. No. 6	Div. No. 7	Div. No. 8	Div. No. 9	Div. No. 10
All.	All	All	All	All	All	All
P_9-P_2	P_9-P_3	P_9-P_4	P_9-P_5&P_4'-P_9'	P_9-P_7&P_1'-P_9'	P_9&P_2-P_9'	P_3-P_9'
A fall	A fall	A rise	A rise	A rise	A rise	A rise
All	All	All	All	All	All	All
All	All	All	All	All	All	All
P_1-P_9'	P_2-P_9'	P_3-P_9'	P_5-P_7'	P_6-P_1'	P_8-P_2	P_9-P_3
A rise	A rise	A fall	A fall	A fall	A fall	A fall
All	All	All	All	All	All	All

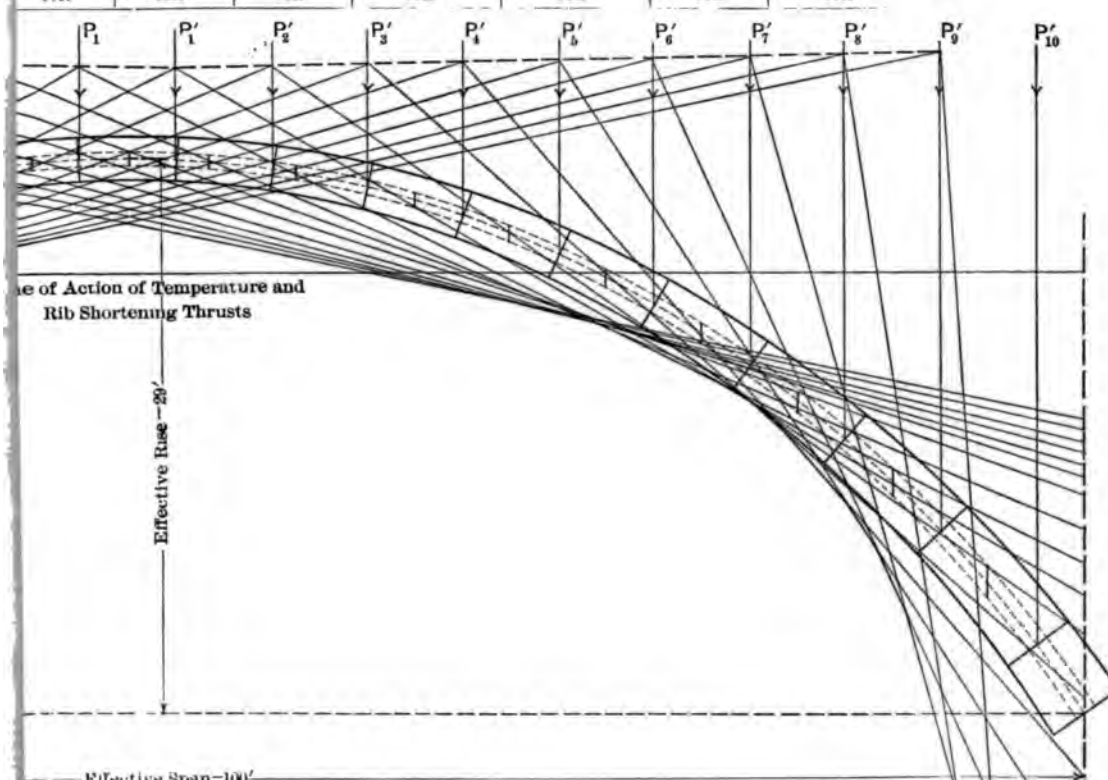


Fig. 37c.

ENCE TABLE FOR MAXIMUM TENSION

P_9-P_2	P_9-P_3	P_9-P_4	P_9-P_6&P_8'-P_9'	P_9-P_7&P_3'-P_9'	P_9&P_1-P_9'	P_2-P_9'
P_1-P_9'	P_2-P_9'	P_3-P_9'	P_4-P_3'	P_6-P_1	P_8-P_2	P_9-P_4

To face p. 98

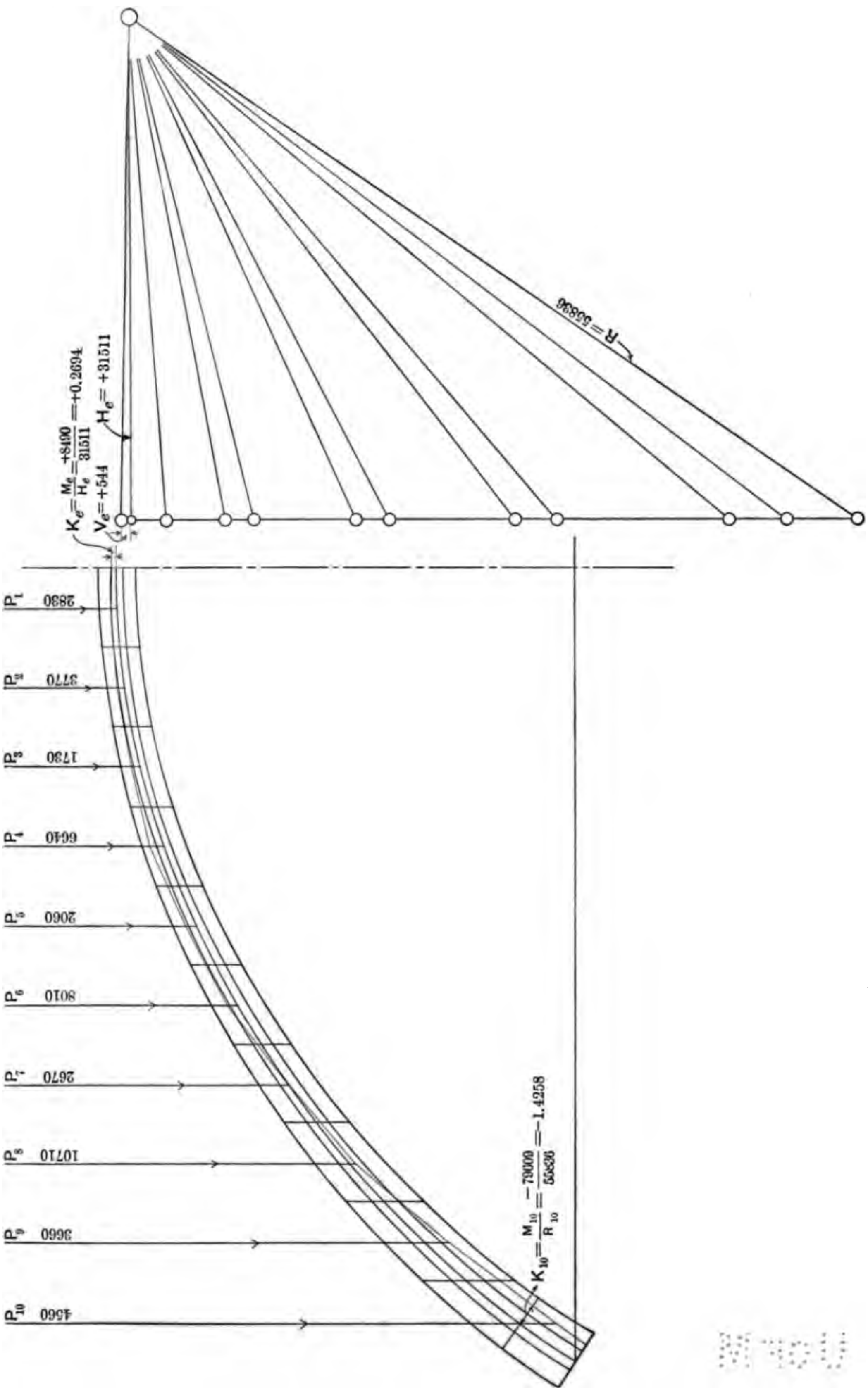

$K_e = \frac{M_e}{H_e} = \frac{+8490}{31511} = +0.2694$
$H_e = +31511$
$V_e = +544$
R = 55836
P_1 2880
P_2 3770
P_3 1730
P_4 6640
P_5 2060
P_6 8010
P_7 2670
P_8 10710
P_9 3660
P_{10} 4560
$K_{10} = \frac{M_{10}}{R_{10}} = \frac{-79009}{55836} = -1.4258$

Fig. 37j.

Compression on lower fiber

$$=\frac{41,400}{2.5}+15,141\frac{1.25}{1.30}$$

$$=16,600+14,600=31,200 \text{ lbs. per sq.ft.} = +220 \text{ lbs. per sq.in.},$$

Tension on upper fiber

$$=16,600-14,600= +2000 \text{ lbs. per sq.ft.} = +15 \text{ lbs. per sq.in.},$$

that is, as the flexural stress on these fibers is less than the direct stress, there is no tension, but a compression of 15 lbs. per sq.in.

In the preceding the resultant of V_4 and H_4 has been used as though it were normal to the radial section through the center of Div. No. 4. This resultant is rarely normal to such a section, but as it is always very nearly so it is exact enough and on the safe side to use it as has been done.

Calculations similar to the preceding should be made for each division and the security of the structure thus determined. The computations may be facilitated by arranging them in tabular form. The influence tables giving loadings producing maximum fiber stress on each division should be made before computing any of the fiber stresses.

Taking each section successively from 1 to 10 the influence tables of Plate I, facing page 98 are prepared. Of these tables:

Group I gives all the loads which are to be considered in computing maximum compression on the upper fibers of any section.

Group II gives the loading to be taken for maximum compression on the lower fibers of any section.

TABLE 37*h*

VALUES OF V_e, H_e, AND M_e FOR THE SEVERAL CONDITIONS OF LOADING

Load at		10	9	8	7	6	5	4	3	2	1	Σ	Σ×2
	Actual Load	4560	3660	9020	2670	6510	2060	5140	1730	3770	2830		
D.L.	V_e		39	288	170	689	326	1137	507	1405	1293		0
	H_e		183	1238	673	2509	1082	3390	1342	3245	2565	16227	32454
	M_e		−627	−3846	−1830	−5670	−1808	−3033	190	5056	9014	−2554	−5108
	Actual Load			1690		1500		1500		1120	750		
L.L.	V_e			54		158		332		417	343	+1304	0
	H_e			232		578		989		964	680	3443	6886
	M_e			−721		−1306		−885		1502	2389	+ 979	+1958

Table No. 37*i*

VALUES OF V, H, AND M AT EACH DIVISION FOR THE LOADINGS OF GROUPS I AND II

Group	Loading	V_{10}	H_{10}	M_{10}	V_9	H_9	M_9	V_8	H_8	M_8	V_7	H_7	M_7	V_6	H_6	M_6
I	D.L.	−41950	+32454	− 2740	−37390	+32454	− 9668	−33730	+32454	+ 5794	−24710	+32454	+ 2580	−22040	+32454	+ 8339
	L.L.	− 2414	+ 5087	+30061	− 2414	+ 5087	+11759	− 2940	+ 3675	+ 7616	− 2668	+ 2609	+ 4110	− 2456	+ 1799	+10759
	Temp.	0	+ 2330	+45455	0	+ 2330	+31498	0	+ 2330	+20407	0	+ 2330	+11272	0	+ 2330	+ 3746
	Rib Sh.	0	− 412	− 8039	0	− 412	− 5571	0	− 412	− 3610	0	− 412	− 1995	0	− 412	− 664
	Total	−44364	+39459	+64737	−39804	+39459	+28018	−36670	+38047	+30207	−27378	+36981	+15967	−24496	+36171	+22180
II	D.L.	−41950	+32454	− 2740	−37390	+32454	− 9668	−33730	+32454	+ 5794	−24710	+32454	+ 2580	−22040	+32454	+ 8339
	L.L.	− 4146	+ 1799	−23375	− 4849	+ 2763	−14449	− 3963	+ 3891	− 9456	− 5028	+ 5844	−17648	− 2414	+ 5087	−12497
	Temp.	0	− 2330	−45455	0	− 2330	−31498	0	− 2330	−20407	0	− 2330	−11272	0	− 2330	− 3746
	Rib Sh.	0	− 412	− 8039	0	− 412	− 5571	0	− 412	− 3610	0	− 412	− 1995	0	− 412	− 664
	Total	−46096	+31511	−79609	−42239	+32475	−61186	−37693	+33603	−27679	−29738	+35556	−28335	−24454	+34799	− 8568
I Comp. on upper fiber	$\frac{\sqrt{V_{10}^2+H_{10}^2}}{A}+\frac{M_{10}c}{I}=$	15300+25800 =41100			15612+13042=28654			15867+16346=32213			14890+10033=24923			15168+16042=31210		
II Comp. on lower fiber	$\frac{\sqrt{V_{10}^2+H_{10}^2}}{A}-\frac{M_{10}c}{I}=$	14390+31730 =46120			14840+28480=43320			15164+14977=30141			15000+17803=32803			14768+ 6197=20965		

Group	Loading	V_5	H_5	M_5	V_4	H_4	M_4	V_3	H_3	M_3	V_2	H_2	M_2	V_1	H_1	M_1
I	D.L.	−15530	+32454	+ 1465	−13470	+32454	+ 3062	− 8330	+32454	− 3301	− 6600	+32454	− 1733	− 2830	+32454	− 2836
	L.L.	− 956	+ 1799	+10854	− 1659	+ 2763	+12469	− 566	+ 3443	+ 9324	− 963	+ 3891	+ 9242	− 418	+ 4277	+ 7219
	Temp.	0	− 2330	+ 2335	0	− 2330	+ 7063	0	− 2330	+10512	0	− 2330	+12772	0	− 2330	+13867
	Rib Sh.	0	− 412	+ 412	0	− 412	+ 1248	0	− 412	+ 1858	0	− 412	+ 2258	0	− 412	+ 2451
	Total	−16486	+31511	+15066	−15129	+32475	+23842	− 8896	+33155	+24995	− 7563	+33603	+22539	− 3248	+33989	+26373
II	D.L.	−15530	+32454	+ 1465	−13470	+32454	+ 3062	− 8330	+32454	− 3301	− 6600	+32454	− 1733	− 2830	+32454	− 2836
	L.L.	− 2414	+ 5087	−13679	− 1711	+ 4123	−12389	− 1304	+ 3443	−10083	− 907	+ 2995	− 7316	− 1082	+ 2609	− 3011
	Temp.	0	+ 2330	− 2335	0	+ 2330	− 7063	0	+ 2330	−10512	0	+ 2330	−12772	0	+ 2330	−13867
	Rib Sh.	0	− 412	+ 412	0	− 412	+ 1248	0	− 412	+ 1858	0	− 412	+ 2258	0	− 412	+ 2451
	Total	−17944	+39459	−14137	−15181	+38495	−15142	− 9634	+37815	−22038	− 7507	+37367	−19563	− 3912	+36981	+17263
I Comp. on upper fiber	$\frac{\sqrt{V_5^2+H_5^2}}{A}+\frac{M_5c}{I}=$	13370+12780 =26150			14330+22890=37220			18270+26485=44755			15038+25782=40820			15640+30702=46342		
II Comp. on lower fiber	$\frac{\sqrt{V_5^2+H_5^2}}{A}-\frac{M_5c}{I}=$	16295+11990 =28285			16552+14538=31090			20402+23350=43752			16640+22378=39018			17180+20095=37275		

Table No. 37j

VALUES OF V, H, AND M AT EACH DIVISION FOR THE LOADINGS OF GROUPS III AND IV

Group	Loading	V_{10}	H_{10}	M_{10}	V_9	H_9	M_9	V_8	H_8	M_8	V_7	H_7	M_7	V_6	H_6	M_6
III	D.L.				−37390	+32454	− 9668	−33730	+32454	+ 5794	−24710	+32454	+ 2580			
	L.L.				− 1711	+ 4123	+12952	− 2597	+ 2995	+ 8067	− 3032	+ 1042	+ 6182			
	Temp.				0	+ 2330	+31498	0	+ 2330	+20407	0	+ 2330	+11272			
	Rib Sh.				0	− 412	− 5571	0	− 412	− 3610	0	− 412	− 1995			
	Total	−44364	+39459	+64737	−39101	+38495	+29211	−36327	+37367	+30658	−27742	+35414	+18039	−24496	+36171	+22180
IV	D.L.				−37390	+32454	− 9668	−33730	+32454	+ 5794	−24710	+32454	+ 2580			
	L.L.				− 4146	+ 1799	−13243	− 3620	+ 3211	− 9002	− 2938	+ 4277	− 8992			
	Temp.				0	− 2330	−31498	0	− 2330	−20407	0	− 2330	−11272			
	Rib Sh.				0	− 412	− 5571	0	− 412	− 3610	0	− 412	− 1995			
	Total	−46096	+31511	−79609	−41536	+31511	−59980	−37350	+32923	−27225	−27648	+33889	−19679	−24454	+34799	− 8568
III Tension on lower fiber	$\frac{\sqrt{V_{10}^2+H_{10}^2}}{A}-\frac{M_{10}c}{I}=$ 15300−2580 = −10500				15423−13600 = +1823			15693−16590 = −897			14486−11406 = +3080			15168−16042 = −874		
IV Tension on upper fiber	$\frac{\sqrt{V_{10}^2+H_{10}^2}}{A}+\frac{M_{10}c}{I}=$ 14390−31730 = −17340				14678−27920 = −13242			15030−14732 = +298			14590−12444 = +2146			14768−6197 = +8571		

Group	Loading	V_5	H_5	M_5	V_4	H_4	M_4	V_3	H_3	M_3	V_2	H_2	M_2	V_1	H_1	M_1
III	D.L.															
	L.L.															
	Temp.															
	Rib Sh.															
	Total	−16486	+31511	+15066	−15129	+32475	+23842	− 8896	+33155	+24995	− 7563	+33603	+22539	− 3248	+33989	+26373
IV	D.L.															
	L.L.															
	Temp.															
	Rib Sh.															
	Total	−17944	+39459	−14137	−15181	+38495	−15142	− 9634	+37815	−22038	− 7507	+37367	−19563	− 3912	+36981	−17263
III Tension on lower fiber	$\frac{\sqrt{V_5^2+H_5^2}}{A}-\frac{M_5c}{I}=$ 13370−12780 = +590				14330−22890 = −8560			18270−26485 = −8215			15038−25782 = −10744			15640−30702 = −15062		
IV Tension on upper fiber	$\frac{\sqrt{V_5^2+H_5^2}}{A}+\frac{M_5c}{I}=$ 16295−11990 = +14305				16552−14538 = +2014			20402−23350 = −2948			16640−22378 = −5738			17180−20095 = −2915		

Group III gives the positions of the live load which are to be used with the other conditions of loading of Group I for maximum tension on the lower fibers of any section.

Group IV gives the positions of the live load which are to be used with the other conditions of loading of Group II for maximum tension on the upper fibers of any section.

Groups I and III, and II and IV are the same except in the positions of live load for a few cases.

A table of values for V_e, H_e and M_e for the actual live and dead loads at each division should next be prepared. This table is No. 37*h* of page 100.

Table No. 37*i* of page 101 gives the values of V, H and M at the centers of each division of the arch ring which must be taken for maximum compressive unit stresses on the *upper* and *lower* fibers of the arch rib at the respective divisions.

Table No. 37*j* of page 102 gives the same quantities for maximum tensile unit stresses on the *lower* and *upper* fibers of the arch rib at the respective divisions.

The maximum fiber stresses in tension and compression may now be determined as is done in the tables of pages 101 and 102. Table No. 37*k* of this page gives a condensed statement of these maximum stresses. This table indicates that the dimensions of the arch ring which was taken for illustration should be slightly increased both at the crown and skewbacks.

Instead of making the analytical computation of pages 101 and 102, the amount and direction of the resultant may be determined from a force polygon and the location from the equilibrium polygon by the well-known methods of graphic statics.

On page 99 Fig. 37*f* is shown the graphical method of determining the resultant for the loading of Group IV which produces maximum tension on the upper fibers of division No. 10.

TABLE No. 37*k*

MAXIMUM COMPRESSION AND TENSION IN LBS. PER SQ.IN.

Division No.	10	9	8	7	6	5	4	3	2	1
Comp. on Upper Fiber........	285	199	224	173	216	182	258	311	283	322
Tension on Upper Fiber.......	120	92						20	40	105
Comp. on Lower Fiber........	320	301	210	228	145	196	216	304	271	259
Tension on Lower Fiber.......	73		6		6		59	57	75	20

ART. 38. ERECTION OF MASONRY OR PLAIN CONCRETE ARCHES

The erection of any statically indeterminate structure is a matter requiring the best engineering judgment. The character of the falsework required to support an arch rib until the rib becomes self-supporting depends in a great degree on the height of the intrados above the ground below and on the character of this earth below. No matter what the form of centering or falsework used, its surface where it comes in contact with and supports the intrados of the arch should be such as to hold the rib so that it may be without initial stress due to erection methods. To secure freedom from initial stress the arch rib may be made in small sections with open joints between the sections. That is like the separate voussoirs of a stone arch. The numerous sections load the falsework for almost the full amount of rib load and produce almost the maximum deflections in the falsework. The joints between the sections should then be filled in as short a time as is possible, and joints symmetrically located with reference to the center filled simultaneously. If the arch rib has been designed to support itself and all subsequent erection operations, the falsework should be taken from under the rib as soon as it is completed and the mortar set, by lowering it as a whole slowly until the rib carries its own load. Most plain concrete or cut-stone masonry arches will have a rib of sufficient dimensions to support itself under its own weight, and under the subsequent loading by the material necessary to complete it; provided the arch ring has been made of proper dimensions to resist the stresses due to full dead load, temperature, rib shortening and live load placed in the most trying positions, and provided the material of the structure other than the rib be placed on the rib in a proper manner as to the order of loading the various points of the rib.

To illustrate the general method to be followed in examining an arch for determining the proper order for the execution of the work subsequent to building the rib, the arch of Fig. 37*a* will be taken, although the investigation of Art. 37 showed that certain of its sections had tensile unit stresses higher than warranted by good practice. This rib will be investigated for twelve conditions of loading or for the following twelve cases.[1]

Case No. 1. Full dead load of the arch rib and a rise of 50° F. in temperature.

Case No. 2. Full dead load of the arch rib and a fall of 50° F. in temperature.

Following the method given in the preceding article, the values of the horizontal and vertical components of, and the moment of the thrust at the crown are

[1] The drawings and computations for much of this article were made at the author's suggestion by his assistants Messrs. Carson and Squire.

TABLE No. 38a

VALUES OF H_e AND M_e FOR DEAD LOAD OF ARCH RING

Div. No.	10	9	8	7	6	5	4	3	2	1	Σ	Σ×2
Load	4560	3660	3080	2670	2330	2060	1860	1730	1620	1590		
H_e		+183	+ 423	+ 673	+ 898	+1082	+1227	+1342	+1394	+1441	+8663	+17236
M_e		−627	−1313	−1830	−2030	−1808	−1097	+ 190	+2173	+5046	−1278	− 2556

NOTE $2\Sigma V_e = 0$.

TABLE No. 38b

VALUES OF V, H, AND M AT EACH DIVISION FOR DEAD LOAD, TEMPERATURE AND RIB SHORTENING

Case	Loading	V_{10}	H_{10}	M_{10}	V_9	H_9	M_9	V_8	H_8	M_8	V_7	H_7	M_7	V_6	H_6	M_6
I	D.L.	−25160	+17326	− 1880	−20600	+17326	− 1848	−16940	+17326	+ 485	−13860	+17326	+ 2233	−11190	+17326	+ 2427
	Temp.	0	+ 2330	+45455	0	+ 2330	+31498	0	+ 2330	+20407	0	+ 2330	+11272	0	+ 2330	+ 3746
	Rib. Sh.	0	− 226	− 4410	0	− 226	− 3056	0	− 226	− 1979	0	− 226	− 1092	0	− 226	− 261
	Total	−25160	+19430	+39165	−20600	+19430	+26594	−16940	+19430	+18913	−13860	+19430	+12413	−11190	+19430	+ 5912
II	D.L.	−25160	+17326	− 1880	−20600	+17326	− 1848	−16940	+17326	+ 485	−13860	+17326	+ 2233	−11190	+17326	+ 2427
	Temp.	0	− 2330	−45455	0	− 2330	−31498	0	− 2330	−20407	0	− 2330	−11272	0	− 2330	− 3746
	Rib. Sh.	0	− 226	− 4410	0	− 226	− 3056	0	− 226	− 1979	0	− 226	− 1092	0	− 226	− 261
	Total	−25160	+14770	−51745	−20600	+14770	−36402	−16940	+14770	−21901	−13860	+14770	−10131	−11190	+14770	− 1580

Case	Loading	V_5	H_5	M_5	V_4	H_4	M_4	V_3	H_3	M_3	V_2	H_2	M_2	V_1	H_1	M_1
I	D.L.	− 8860	+17326	+ 1613	− 6800	+17326	+ 510	− 4940	+17326	− 428	− 3210	+17326	− 1151	− 1590	+17326	− 1343
	Temp.	0	− 2330	+ 2335	0	− 2330	+ 7063	0	− 2330	+10512	0	− 2330	+12772	0	− 2330	+13867
	Rib. Sh.	0	− 226	+ 229	0	− 226	+ 690	0	− 226	+ 1024	0	− 226	+ 1242	0	− 226	+ 1349
	Total	− 8860	+14770	+ 4177	− 6800	+14470	+ 8263	− 4940	+14770	+11108	− 3210	+14770	+12863	− 1590	+14770	+13873
II	D.L.	− 8860	+17326	+ 1613	− 6800	+17326	+ 510	− 4940	+17326	− 428	− 3210	+17326	− 1151	− 1590	+17326	− 1343
	Temp.	0	+ 2330	− 2335	0	+ 2330	− 7063	0	+ 2330	−10512	0	+ 2330	−12772	0	+ 2330	−13867
	Rib. Sh.	0	− 226	+ 229	0	− 226	+ 690	0	− 226	+ 1024	0	− 226	+ 1242	0	− 226	+ 1349
	Total	− 8860	+19430	− 493	− 6800	+19430	− 5863	− 4940	+19430	− 9916	− 3210	+19430	−12681	− 1590	+19430	−13861

given in Table No. 38*a* for the dead load only. Table No. 38*b* gives the vertical and horizontal components of the thrust and its moment at the center of each division of the rib.

Table No. 38*c* gives the compressive and tensile unit stresses on the extreme fibers of the sections of the rib through the centers of each division of the rib.

Table No. 38*c*

COMPRESSION AND TENSION IN ARCH RING FOR CASES NOS. 1 AND 2

Case No. 1

Division.	$\frac{\sqrt{V^2+H^2}}{A}$	$\frac{Mc}{I}$	Compression on Upper Fiber.		Tension on Lower Fiber.	
			Lbs. per Sq.ft.	Lbs. per Sq.in.	Lbs. per Sq.ft.	Lbs. per Sq.in.
10	8194	15608	23802	165	7414	51
9	7888	12382	20270	140	4494	31
8	7741	10236	17977	125	2495	17
7	7724	7800	15524	108	76	1
6	7786	4276	12062	84	0	0
5	8028	3543	11571	80	0	0
4	8234	7934	16168	112	0	0
3	8424	11770	20194	140	3346	23
2	8600	14714	23314	162	6114	42
1	8588	16152	24740	172	7564	53

Case No. 2

Division.	$\frac{\sqrt{V^2+H^2}}{A}$	$\frac{Mc}{I}$	Compression on Lower Fiber.		Tension on Upper Fiber.	
			Lbs. per Sq.ft.	Lbs. per Sq.in.	Lbs. per Sq.ft.	Lbs. per Sq.in.
10	7520	20622	28142	196	13102	91
9	7061	16947	24008	167	9886	69
8	6750	11852	18602	129	5102	35
7	6556	6366	12922	90	0	0
6	6434	1143	7577	53	0	0
5	6476	418	6884	48	0	0
4	6504	5629	12133	84	0	0
3	6545	10507	17052	118	3962	28
2	6601	14506	21107	147	7905	55
1	6545	16138	22683	157	9593	67

Fig. 38*a* shows the force and equilibrium polygon drawings for the loadings of Cases No. 1 and 2. Ke', K_{10}', He' and R_{10}' are for a rise in temperature; and Ke'', K_{10}'', He'' and R_{10}'' are for a fall in temperature. Table No. 38*c* and the equilibrium polygon drawing show very clearly the variation of the stress distribution. The compressive unit stresses are all small and need no comment.

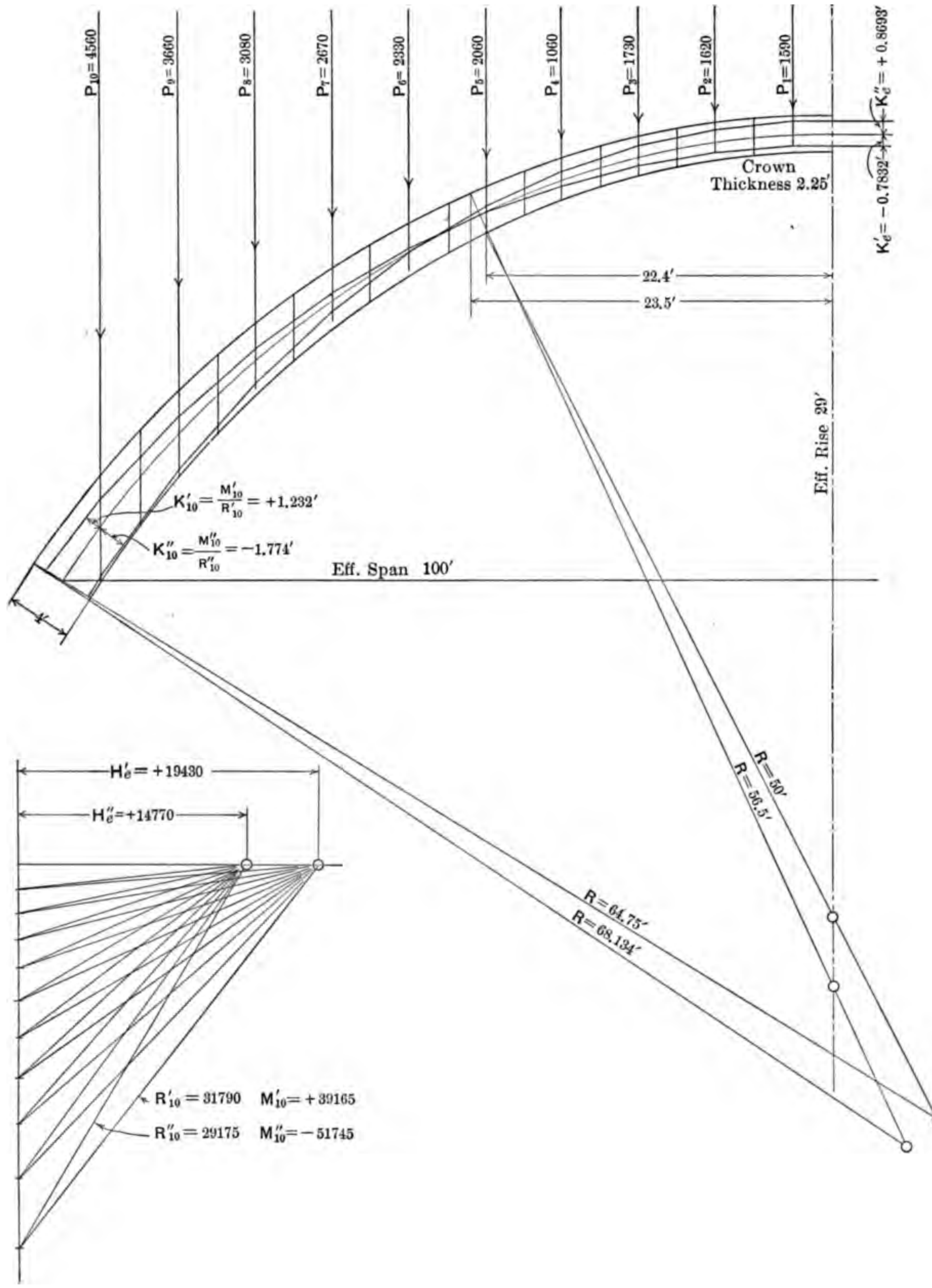

P10=4560
P9=3660
P8=3080
P7=2670
P6=2330
P5=2060
P4=1060
P3=1730
P2=1620
P1=1590
K'c=-0.7832'
K''c=+0.8693'
Crown
Thickness 2.25'
22.4'
23.5'
Eff. Rise 29'
K'10=M'10/R'10=+1.232'
K''10=M''10/R''10=-1.774'
Eff. Span 100'
H'c=+19430
H''c=+14770
R=50'
R=56.5'
R=64.75'
R=68.134'
R'10=31790 M'10=+39165
R''10=29175 M''10=-51745

Fig. 38a.

The maximum tension is 91 pounds per sq.in. at the upper extreme fiber of division No. 10, which is too high and should not be permitted. It should be remembered that the investigation of the previous article showed even higher values of the tensile unit stress for this rib and that therefore if the accepted design of a rib has unit stresses within allowable limits for the conditions of loading possible when in use, it will generally be of ample dimensions to permit all erection operation subsequent to that of building the rib, as has been before stated. A range of temperature throughout the mass of material of, and during the limited time required to construct the arch rib would probably not exceed ±20° F. Therefore the investigation for Cases No. 3 and 4, which are the same as Cases No. 1 and 2, except that temperature variation is from +20 to −20° F. is made. Tables No. 38*d* and 38*e* give the vertical and horizontal components of the thrust, and its moment at the crown and division No. 10.

TABLE No. 38*d*

	Case No. 3			Case No. 4		
	V_c'	H_c'	M_c'	V_c''	H_c''	M_c''
D.L.	0	+17326	−2556	0	+17326	−2556
Temp.	0	+ 930	−5610	0	− 930	+5610
Rib Sh.	0	− 226	+1365	0	− 226	+1365
Total	0	+18030	−6801	0	+16170	+4419
$K'_c=\dfrac{-6801}{+18030}=-0.3774'$				$K_c''=\dfrac{+4419}{+16170}=+0.2733$		

TABLE No. 38*e*

	Case No. 3			Case No. 4		
	V_{10}'	H_{10}'	M_{10}'	V_{10}''	H_{10}''	M_{10}''
D.L.	+25160	+17326	− 1880	+25160	+17326	− 1880
Temp.	0	+ 930	+18130	0	− 930	−18130
Rib Sh.	0	− 226	− 4410	0	− 226	− 4410
Total	+25160	+18030	+11820	+25160	+16170	−24420
$K_{10}'=\dfrac{M_{10}'}{R_{10}'}=\dfrac{+11820}{+30950}=+0.3819$				$K_{10}''=\dfrac{-24420}{+29910}=-0.8164$		

Fig. 38*b* shows the forces and equilibrium polygons; and, comparing them with Fig. 38*a*, the effect of the smaller range in temperature is shown. The limit of the middle third lies 0.375 foot above and 0.375 foot below the axis at the crown, and

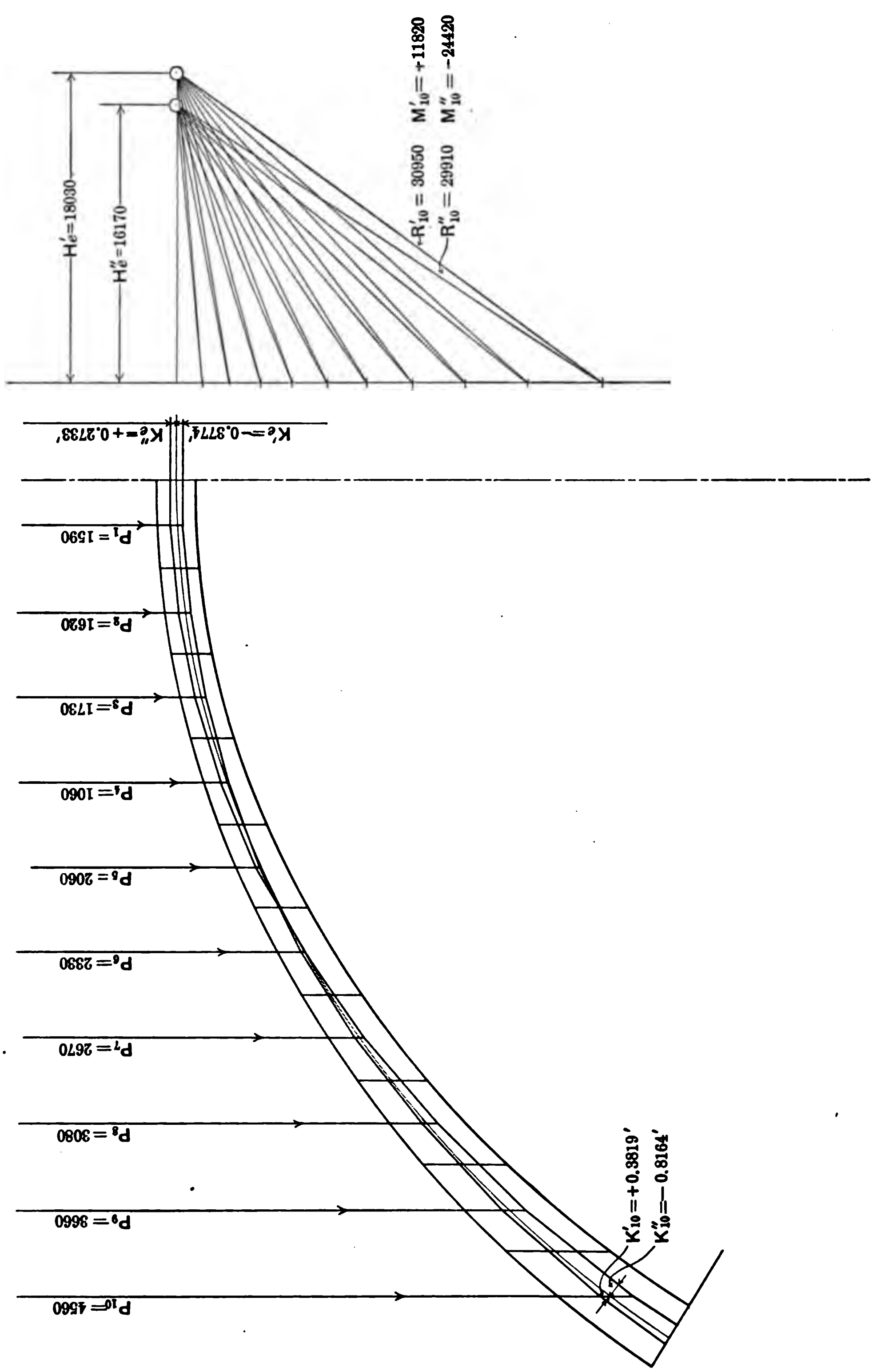

H'e=18030
H''e=16170
R'10 = 30950 M'10 = +11820
R''10 = 29910 M''10 = -24420
K'e = -0.3774'
K''e = +0.2733'
P1 = 1590
P2 = 1620
P3 = 1730
P4 = 1060
P5 = 2060
P6 = 2330
P7 = 2670
P8 = 3080
P9 = 3660
P10 = 4560
K'10 = +0.3819'
K''10 = -0.8164'

Fig. 38b.

0.6467 foot above and 0.6467 foot below the axis at division No. 10. If the values of K, which is the distance of the resultant from the axis at any point, are within these limits, no tension will be produced on the arch ring. As K_e' is only 0.0024 foot greater than the distance to the limit of the middle third, it will not be necessary to consider the small tension here. K_{10}'' is 0.1697 foot greater than the middle third distance and accordingly the stress for the case will be computed as follows:

$$R_{10}'' = \sqrt{(25{,}160)^2 + (16{,}170)^2} = 29{,}910,$$

$$\frac{R_{10}''}{A} = \frac{29{,}910}{3.88} = 7{,}710 \text{ lbs. per sq.ft.}$$

$$\frac{M_{10}''c}{I} = \frac{24{,}420 \times 1.94}{4.868} = 9730 \text{ lbs. per sq.ft.}$$

$7710 + 9730 = 17{,}440$ lbs. per sq.ft. $= 121$ lbs. sq.in. compression on lower fiber.

$7710 - 9730 = -2020$ lbs. per sq.ft. $= 14$ lbs. per sq.in. tension on upper fiber.

From the equilibrium polygons it is seen that the resultant pressure lines tend toward the axis as they approach divisions Nos. 5 and 6 from the crown, and from division No. 10. It is seen for these cases, therefore, that the arch ring is not stressed beyond the allowed limit and is capable of standing alone.

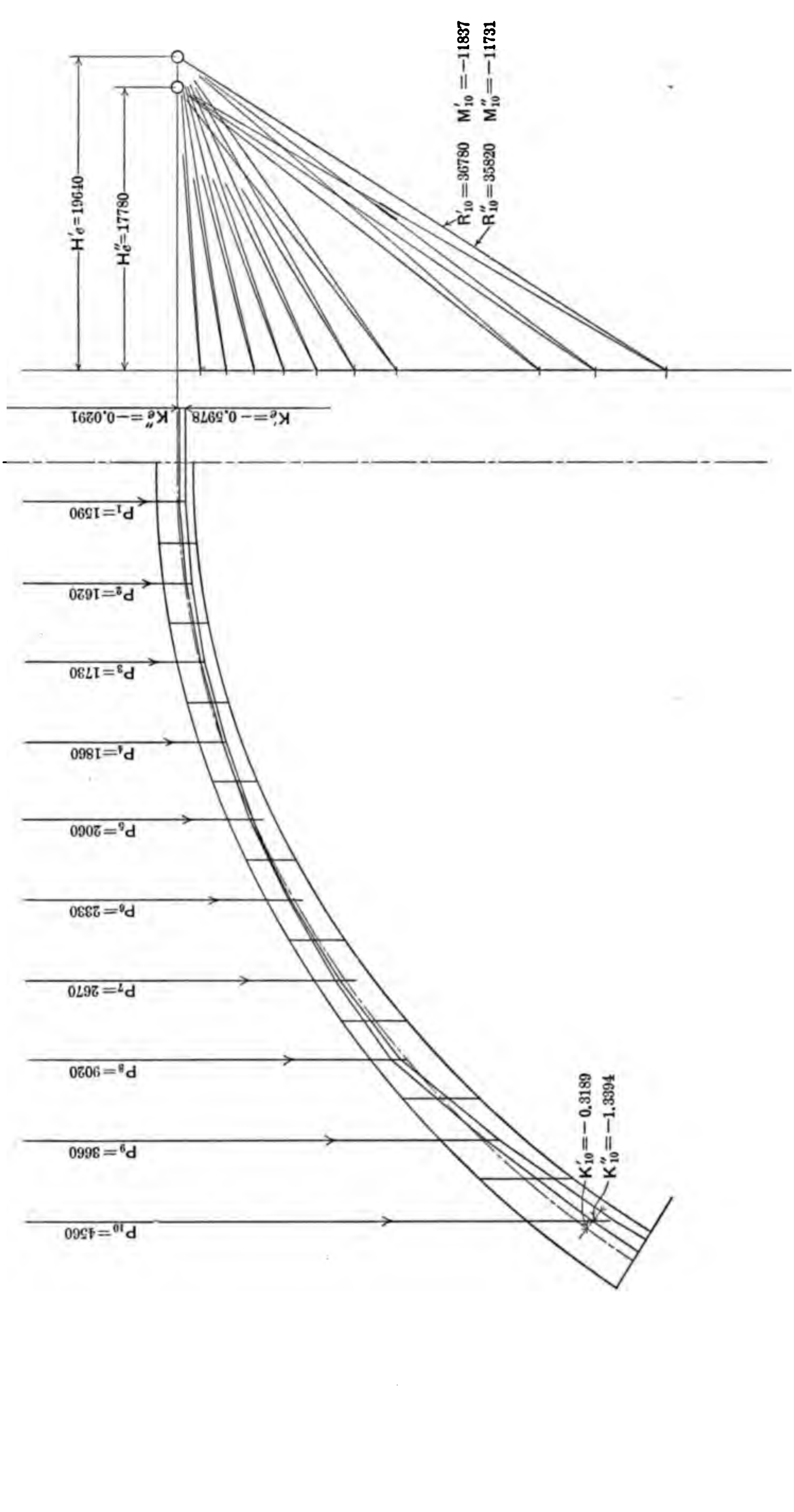
Cases No. 5 and 6 are the same as Cases No. 3 and 4, with the addition of a full dead load at divisions No. 8 and 8′. Table No. 38*f* gives amount of the thrust

TABLE No. 38*f*

DIVISION LOADS FOR CROSS WALLS AND FLOOR WITH THEIR RESULTING CROWN THRUST AND MOMENT

Div. No.	8	6	4	2	1
Load	5,940	4,180	3,280	2,150	1,240
H_e	+ 816	+1,611	+2,163	+1,850	+1,124
M_e	−2,533	−3,641	−1,935	+2,883	+3,950

and moment at the crown due to dead loads of cross walls and floor. Tables No. 38*g* and 38*h* give, for Cases No. 5 and 6, the amounts of the vertical and horizontal components of the thrust, and the moment of the thrust at the crown and division No. 10. Fig. 38*c* shows the equilibrium polygons for this condition of loading. The loading of Case 5 gives the maximum tension at the crown, and the loading of Case 6 gives the maximum tension at No. 10. With the quantities of



H'c=19640
H''c=17780
R'10=36780 M'10=−11837
R''10=35820 M''10=−11731
K'c=−0.5978
K''c=−0.0291
P1=1590
P2=1620
P3=1730
P4=1860
P5=2060
P6=2330
P7=2670
P8=3020
P9=3660
P10=4560
K'10=−0.3189
K''10=−1.3394

Fig. 38c.

TABLE No. 38g

	Case No. 5			Case No. 6		
	V_e'	H_e'	M_e'	V_e''	H_e''	M_e''
D.L.......	0	+18,958	− 7,622	0	+18,958	−7,622
Temp.....	0	+ 930	− 5,610	0	− 930	+5,610
Rib Sh....	0	− 248	+ 1,494	0	− 248	+1,494
Total....	0	+19,640	−11,738	0	+17,780	− 518
	$K_e' = \frac{-11,738}{19,640} = -0.5978$			$K_e'' = \frac{-518}{17,780} = -0.0291$		

TABLE No. 38h

	Case No. 5			Case No. 6		
	V_{10}'	H_{10}'	M_{10}'	V_{10}''	H_{10}''	M_{10}''
D.L.......	31,100	+18,958	−25,023	31,100	+18,958	−25,023
Temp.....	0	+ 930	+18,130	0	− 930	−18,130
Rib Sh....	0	− 248	− 4,838	0	− 248	− 4,838
Total....	31,100	+19,640	−11,731	31,100	+17,780	−47,991
	$K_{10}' = \frac{-11,731}{36,780} = -0.3189$			$K_{10}'' = \frac{-47,991}{35,820} = -1.3394$		

Tables No. 38g and 38h the maximum stress intensities at the crown and center of division No. 10 are computed as follows:

$$\frac{H_e'}{A}+\frac{M_e'c}{I}=\frac{19,640}{2.25}+\frac{11,738}{0.8437}=8730+13,910=22,640 \text{ lbs. per sq.ft.,}$$

which equals 157 lbs. per sq.in. in compression in lower fiber.

$$\frac{H_e'}{A}-\frac{M_e'c}{I}=8730-13,910=-5180 \text{ lbs. per sq.ft.,}$$

which equals 36 lbs. per sq.in. tension in upper fibre.

$$\frac{R_{10}''}{A}+\frac{M_{10}''c}{I}=\frac{35,820}{3.88}+\frac{47,991\times 1.94}{4.868}=9230+19,124=28,354 \text{ lbs. per sq.ft.,}$$

which equals 197 lbs. per sq.in. compression in lower fiber.

$$\frac{R_{10}''}{A}-\frac{M_{10}'c}{I}=9230-19,124=-9894 \text{ lbs. per sq.ft.,}$$

which equals 69 lbs. per sq.in. in tension in upper fiber.

The tension at the crown is within the limit allowed; but, at division No. 10 on the upper fiber, there are 69 lbs. per square inch, which is a little too high. Therefore, there must be found some place on the arch where a load or loads can be placed to reduce this tension and not increase the tension at the crown beyond 50 lbs. per square inch. By placing a full dead load at divisions No. 6 and 6′, in addition to the loading of Cases No. 5 and 6, it was found that the tension at the crown increased to 84 lbs. per square inch on the upper fiber. By placing full dead loads at 4 and 4′ in addition to those loads already mentioned, the tension at the crown increased to 100 lbs. per square inch. The influence table may be used for the purpose of determining the kind of stress produced at any point in the arch ring by a load at any division; this table shows that loads at 8, 8′, 6, 6′, and 4, 4′ all produce tension on the upper fiber at the crown, while loads at 2, 2′; 1 and 1′ produce tension on the lower fibre at the crown. To decrease the tension on the upper fiber at division 10, therefore, we must place loads at 2, 2′; 1 and 1′ as indicated by the influence table and diagram of the previous article.

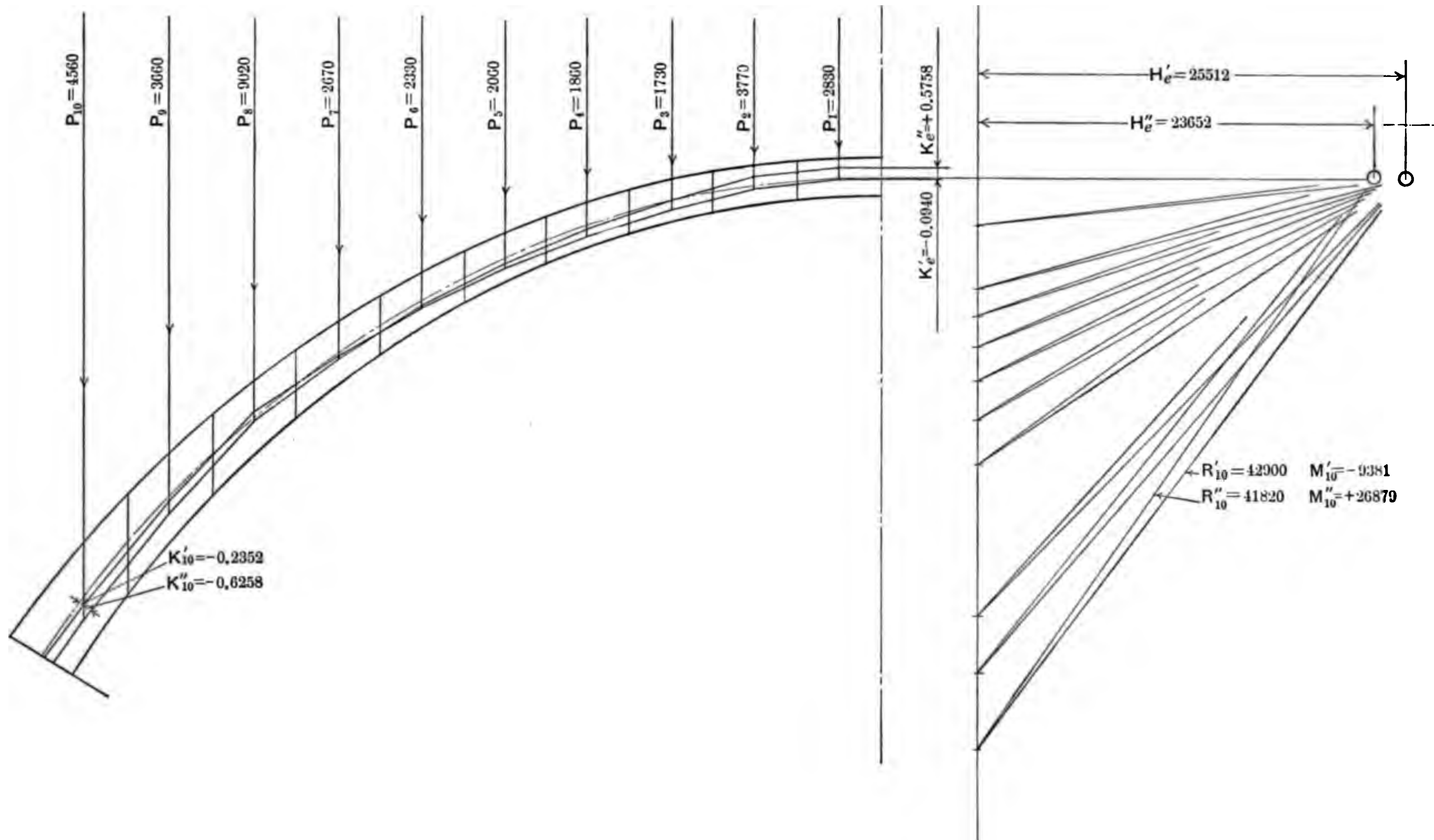
Cases No. 7 and 8 are for the same condition of loading as Cases No. 5 and 6 with addition of full dead loads at divisions Nos. 2, 2′, 1 and 1′. Tables No. 38*i*

TABLE No. 38*i*

	CASE No. 7			CASE No. 8		
	V_e'	H_e'	M_e'	V_e''	H_e''	M_e''
D.L......	0	+24,908	+6,044	0	+24,908	+ 6,044
Temp.....	0	+ 930	−5,610	0	− 930	+ 5,610
Rib Sh.....	0	− 326	+1,963	0	− 326	+ 1,963
Total....	0	+25,512	+2,397	0	+23,652	+13,617
	$K_e' = \frac{+2,397}{25,512} = +0.0940$			$K_e'' = \frac{+13,617}{23,652} = +0.05758$		

TABLE No. 38*j*

	CASE No. 7			CASE No. 8		
	V_{10}'	H_{10}'	M_{10}'	V_{10}''	H_{10}''	M_{10}''
D.L.......	34,490	+24,908	− 1,677	34,490	+24,908	− 1,677
Temp.....	0	+ 930	+18,130	0	− 930	−18,130
Rib Sh....	0	− 326	− 6,361	0	− 326	− 6,361
Total....	34,490	+25,512	+10,092	34,490	+23,652	−26,168
	$K_{10}' = \frac{+10,092}{42,900} = +0.2352$			$K_{10}'' = \frac{-26,168}{41,820} = -0.6258$		



$P_{10}=4560$
$P_9=3660$
$P_8=9020$
$P_7=2670$
$P_6=2330$
$P_5=2060$
$P_4=1860$
$P_3=1730$
$P_2=3770$
$P_1=2880$
$K_e''=+0.5758$
$K_e'=-0.0940$
$H_e'=25512$
$H_e''=23652$
$R_{10}'=42900$ $M_{10}'=-9381$
$R_{10}''=41820$ $M_{10}''=+26879$
$K_{10}'=-0,2352$
$K_{10}''=-0,6258$

FIG. 38d.

and 38*j* give vertical and horizontal components of the thrust, and its moment for the crown, and division No. 10. Fig. 38*d*, shows the force and equilibrium polygons for the loadings of Cases No. 7 and 8. The loading of Case No. 8 produces tension at the crown and no tension at division No. 10, while that for Case No. 7 gives no tension at either place.

For the loading of Case No. 8, the compression in upper fiber

$$=\frac{23,652}{2.25}+\frac{13,617}{0.8437}=10,510+16,140=26,650 \text{ lbs. per sq.ft.},$$

which equals 185 lbs. per square inch, and the tension in the lower fiber

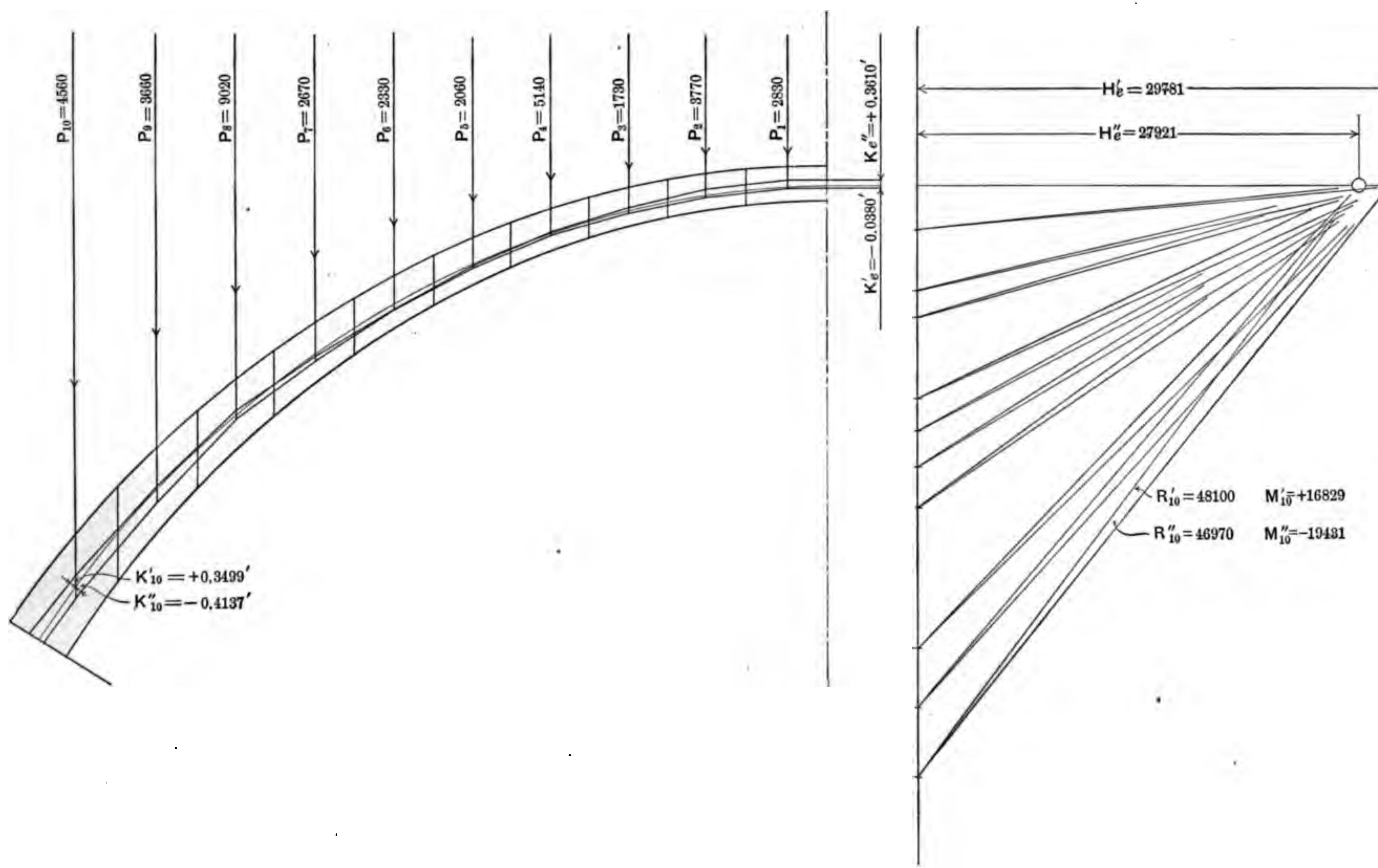
$$=10,510-16,140=-5630 \text{ lbs. per sq.ft.},$$

which equals 39 lbs. per square inch. A tension of 39 lbs. per square inch on the lower fiber at the crown is within the limit set and, therefore, this condition of loading proves satisfactory. Since the loading of Cases No. 5 and 6 on investigation is shown to give too much tension, the proper way to build the arch would be to place loads at divisions 8, 8′, 2, 2′, 1 and 1′ simultaneously, or parts of each of these loads alternately.

The arch rib must be further investigated to see how the remaining loads may be placed in order to finish the bridge. Therefore the investigation is made for the loading of Cases No. 9 and 10, which are the same as Cases No. 7 and 8, with the addition of full dead loads at divisions Nos. 4 and 4′. Tables No. 38*k* and 38*l* give the vertical and horizontal components of the thrust, and its moment at the crown and division No. 10. Fig. No. 38*e* shows the force and equilibrium polygons for this loading. The resultant pressure lines for these cases fall within the middle thrust at each division; and, therefore, since there is no tension in the arch ring these conditions of loading are permissible.

TABLE No. 38*k*

	Case No. 9			Case No. 10		
	V_e'	H_e'	M_e'	V_e''	H_e''	M_e''
D.L.	0	+29,234	+ 2,174	0	+29,234	+ 2,174
Temp.	0	+ 930	− 5,610	0	− 930	+ 5,610
Rib Sh.	0	− 383	+ 2,304	0	− 383	+ 2,304
Total	0	+29,781	− 1,132	0	+27,921	+10,088
	$K_e'=\frac{-1,132}{+29,781}=-0.0380$			$K_e''=\frac{+10,088}{+27,921}=+0.3610$		



P₁₀=4560
P₉=3660
P₈=9020
P₇=2670
P₆=2330
P₅=2060
P₄=5140
P₃=1730
P₂=3770
P₁=2830
K″e=+0,3610′
K′e=−0,0380′
K′₁₀=+0,3499′
K″₁₀=−0,4137′
H′e=29781
H″e=27921
R′₁₀=48100
M′₁₀=+16829
R″₁₀=46970
M″₁₀=−19431

FIG. 38e.

TABLE No. 38l

	Case No. 9			Case No. 10		
	V_{10}'	H_{10}'	M_{10}'	V_{10}''	H_{10}''	M_{10}''
D.L.......	37,770	+29,234	+ 6,173	37,770	+29,234	+ 6,173
Temp.....	0	+ 930	+18,130	0	− 930	−18,130
Rib Sh....	0	− 383	− 7,474	0	− 383	− 7,474
Total. ..	38,770	+29,781	+16,829	37,770	+27,921	−19,431
	$K_{10}'=\frac{+16,829}{48,100}=+0.3499$			$K_{10}''=\frac{-19,431}{46,970}=-0.4137$		

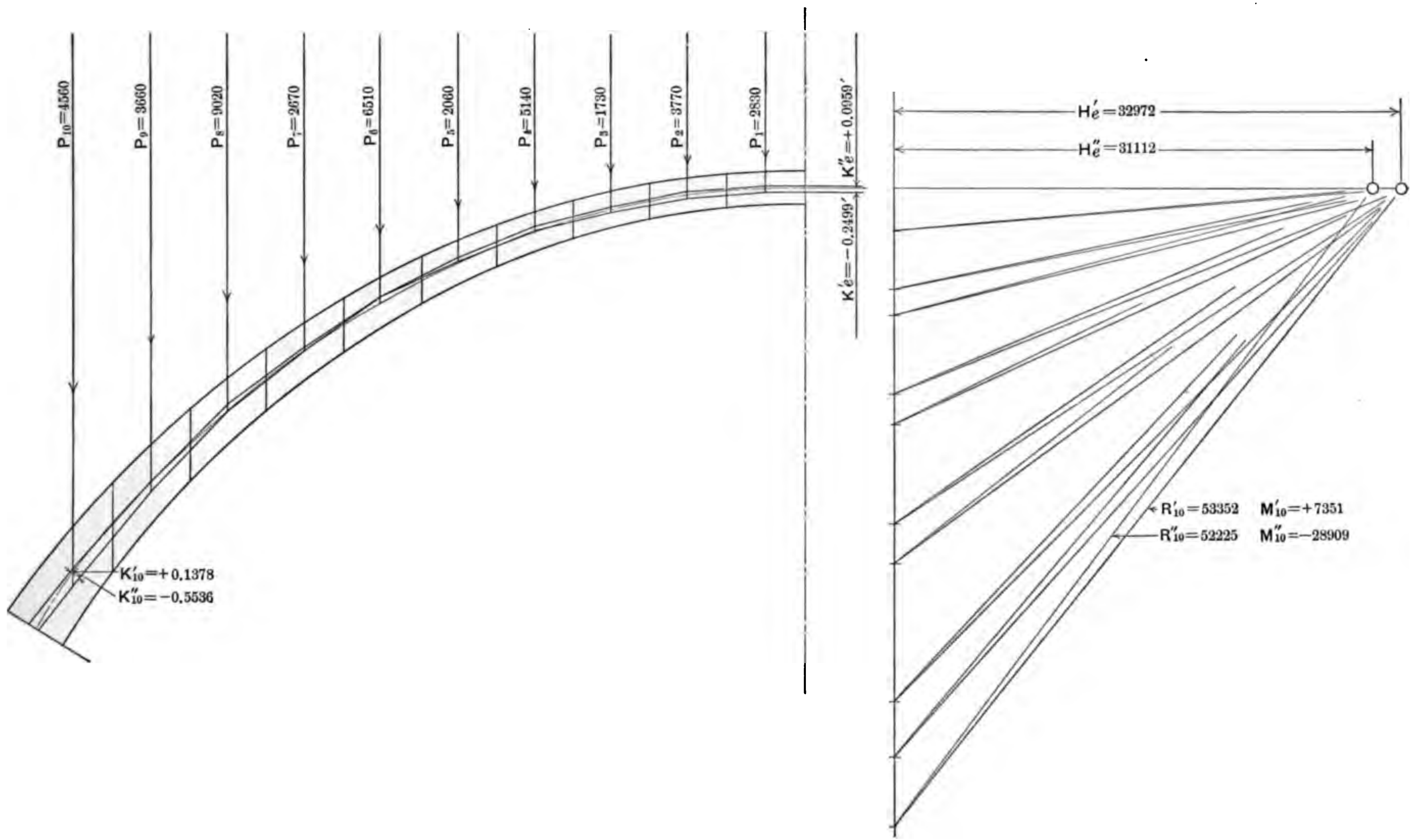
The loadings for Cases No. 11 and 12 are the same as for Cases 9 and 10 with the addition of full dead loads at divisions No. 6 and 6′, or for full dead load of the arch. Tables No. 38m and 38n give the vertical and horizontal components of the thrust, and its moment at the crown and division 10. Fig. 38f shows the force

TABLE No. 38m

	Case No. 11			Case No. 12		
	V_e'	H_e'	M_e'	V_e''	H_e''	M_e''
D.L.......	0	+32,454	−5,108	0	+32,454	−5,108
Temp.....	0	+ 930	−5,610	0	− 930	+5,610
Rib Sh....	0	− 412	+2,480	0	− 412	+2,480
Total....	0	+32,972	−8,238	0	+31,112	+2,982
	$K_e'=\frac{-8,238}{+32,972}=-0.2499$			$K_e''=\frac{+2,982}{+31,112}=+0.0959$		

TABLE No. 38n

	Case No. 11			Case No. 12		
	V_{10}'	H_{10}'	M_{10}''	V_{10}''	H_{10}''	M_{10}''
D.L.......	41,950	+32,454	− 2,740	41,950	+32,454	− 2,740
Temp.....	0	+ 930	+18,130	0	− 930	−18,130
Rib Sh....	0	− 412	− 8,039	0	− 430	− 8,039
Total....	41,950	+32,972	+ 7,351	41,950	+31,112	−28,909
	$K_{10}'=\frac{+7,351}{+53,352}=+0.1378$			$K_{10}''=\frac{-28,909}{+52,225}=-0.5536$		



$P_{10}=4560$
$P_9=3660$
$P_8=9020$
$P_7=2670$
$P_6=6510$
$P_5=2060$
$P_4=5140$
$P_3=1730$
$P_2=3770$
$P_1=2830$
$K''_e=+0.0959'$
$K'_e=-0.2499'$
$K'_{10}=+0.1378$
$K''_{10}=-0.5536$
$H'_e=32972$
$H''_e=31112$
$R'_{10}=53352$ $M'_{10}=+7351$
$R''_{10}=52225$ $M''_{10}=-28909$

FIG. 38*f*.

and equilibrium polygons for the same loadings. The pressure line for both cases lies in the middle third; therefore there is no tension on the arch ring and a safe order for the placing of load on the arch rib is established.

In the foregoing investigation it has been shown that it is possible to build the cross walls and floor system of the arch upon the arch ring after the centering under the latter has been removed, and that the way to do this is to construct the walls and floor at 8, 8′, 2, 2′, 1 and 1′ simultaneously, and then to construct those at 4 and 4′ and finally at 6 and 6′. If this method of applying the dead loads is followed, the tension in the arch ring will always be under the allowable amount. While for arches of a different form, either for the arch axis or the ring, the previous figures would not apply, it can be safely stated that the foregoing gives about the order in which the loading should be applied to any arch rib of plain concrete or masonry.

ART. 39. THE TWO-HINGED ARCH RIB WITH SOLID WEB

The two-hinged arch with a solid web is a structure of such practical utility and beautiful appearance that it is frequently built, and therefore a complete examination of the stresses in such a bridge will be made. This form of arch is generally built with a steel-plate girder rib. Let the full line of Fig. 39*a* show the

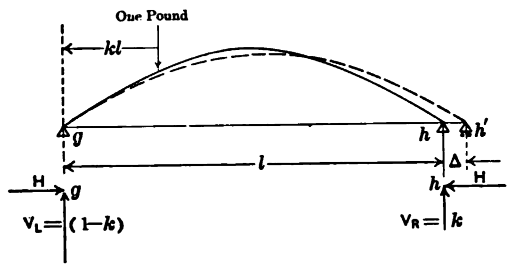


FIG. 39*a*.

axis of such a rib in which at the points g and h there are supposed to be frictionless pins fixed in position. It is clear that, for a vertical load of 1 lb. acting as shown and the support at the right end permitted to move horizontally, a motion to some new position h' will occur. It is certain, therefore, that a horizontal force acting to the left exists at h when the right support is not permitted to move horizontally, and as $\Sigma H = 0$, and as there are no horizontal loads there must be an equal and opposite horizontal force acting to the right at g the left end. It is also certain that the vertical forces (the vertical components of the reactions) at g and h are the same as would exist for a simple beam for a span l, for with the center of

moments at either end the moment of the horizontal forces at the ends would be zero, as their lever arms are zero, and the moment of the load must be equal and opposite to the moment of the reaction at the other end for $\Sigma M=0$ for equilibrium. The nature of the reactions at g and h for vertical loads are as shown just below the supports in Fig. 39a. A method of determining the horizontal components of the reactions will now be developed. For this purpose let Fig. 39b represent the same arch rib as Fig. 39a under the action of a force of 1 lb. acting to the right at h and the support at h free to move horizontally. Then for the point h free to move.

Let Δ be the horizontal deflection of h due to a vertical load of unity as shown in Fig. 39a.

Let d be the horizontal deflection of h due to a horizontal load of unity at h as shown in Fig. 39b.

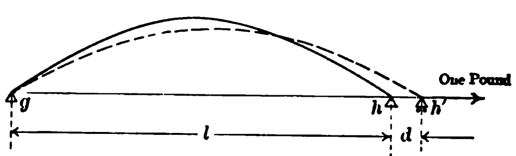


FIG. 39b.

Then for the point h fixed in position:

Let H be the horizontal force developed at h by the vertical load of 1 lb. placed at any point on the rib axis.

The following equation may now be written: $Hd=\Delta$, or $H=\frac{\Delta}{d}$.

The equations for the value of H due to temperature changes and rib shortening may be written from analogy to the foregoing. With all the outer forces known the rib may be investigated as for any statically determined structure.

The computations of the maximum stress in a two-hinged arch rib requires the computation of the deflections Δ and d, and the determination of the location of the given loading to produce these maximum stresses at any point.

To illustrate a method to be followed let Fig. 39c show the general features of the problem. Computation of the weight of the quantities of steel in the spandrel posts, stringers, and floorbeams, together with the other flooring material, is first made and then an estimate of the weight of a rib made. The weight of such a rib should not exceed three-fourths of the weight of a simple span truss to do the same duty as the rib. For this case such computation shows that the dead panel load will be about 8600 lbs. The live load is known to be 30,000 lbs. per panel. In order to make a first assumption of the size of rib to use, compute the

maximum bending moment for full live and dead loads at the center of the span. This moment for this case $=[6\times 36{,}800\times 6-36{,}800(.5+1.5+2.5+3.5+4.5+5.5)]12$ $=18\times 12\times 36{,}800=7{,}948{,}800$ ft.lbs. For a three-hinged arch the thrust would be this moment divided by the rise, for the two-hinged arch the thrust will be taken the same as the thrust for the three-hinged arch. In addition, there is generally a moment at any cross-section which may be assumed as one-tenth

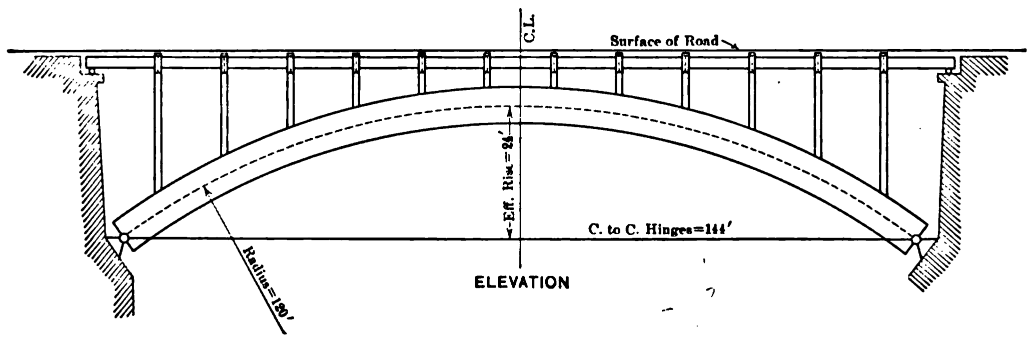


FIG. 39c.

of the center moment for full loading. The trial area A of the rib then equals $\frac{1}{S}\left(T+M\frac{c}{r^2}\right)$ where

S = allowable unit stress = 10,000 for this example;
M = the bending moment;
T = the thrust;
c = the distance from the neutral axis to the extreme fiber;
r = the radius of gyration of the section about the rib axis

For this case

$$A=\frac{1}{10{,}000}\left(\frac{7{,}948{,}800}{24}+\frac{7{,}948{,}800\times 12}{10}\times\frac{c}{r^2}\right).$$

For an I section

$$r=.35\times\text{the depth of the rib}=.7c \text{ approximately,}$$

whence

$$r^2=\frac{c^2}{2}, \text{ approximately,}$$

whence

$$A=\frac{1}{10{,}000}\left(331{,}200+\frac{19{,}077{,}120}{c}\right).$$

From which expression it is seen that the area of the rib depends very largely on the depth of the rib, the deeper the rib the less the material required within limits. The depth of rib should not be so great as to make excessive stiffening of the web plate of the rib necessary.

TABLE No. 39a

COMPUTATIONS FOR d AND Δ

Sect.	Values at Center of Section			ds	$\frac{mds}{I}$	$d=\frac{m^2ds}{I}$	M for 1 lb. at Center of Section No.						$\Delta=Mmds/I$ for 1 lb. at Center of Section No.					
	X	$m=Y$	I				1	2	3	4	5	6	1	2	3	4	5	6
1	6	4.22	3.635	14.39	16.71	70.52	5.75	5.25	4.75	4.25	3.75	3.25	96.14	87.78	79.42	71.06	62.70	54.34
2	18	11.16	3.635	13.44	41.27	460.60	5.25	15.75	14.25	12.75	11.25	9.75	216.72	650.16	588.24	526.32	464.40	402.48
3	30	16.41	3.635	12.84	57.97	951.30	4.75	14.25	23.75	21.25	18.75	16.25	275.34	826.02	1376.74	1231.82	1086.90	941.98
4	42	20.19	3.635	12.40	68.88	1390.70	4.25	12.75	21.25	29.75	26.25	22.75	292.72	878.16	1463.60	2049.04	1807.80	1566.76
5	54	22.64	3.635	12.14	75.62	1712.00	3.75	11.25	18.75	26.25	33.75	29.25	283.58	850.74	1417.90	1985.06	2552.22	2211.69
6	66	23.85	3.635	12.00	78.74	1878.00	3.25	9.75	16.25	22.75	29.25	35.75	255.90	767.70	1279.50	1791.30	2303.10	2814.90
6′	78	23.85	3.635	12.00	78.74	1878.00	2.75	8.25	13.75	19.25	24.75	30.25	216.52	649.56	1082.60	1515.64	1948.68	2381.72
5′	90	22.64	3 635	12.14	75.62	1712.00	2.25	6.75	11.25	15.75	20.25	24.75	170.13	510.39	850.65	1190.91	1531.17	1871.43
4′	102	20.19	3.635	12.40	68.88	1390.70	1.75	5.25	8.75	12.25	15.75	19.25	120.52	361.56	602.60	843.64	1084.68	1325.72
3′	114	16.41	3.635	12.84	57.97	951.30	1.25	3.75	6.25	8.75	11.25	13.75	72.46	217.38	362.30	507.22	652.14	797.06
2′	126	11.16	3.635	13.44	41.27	460.60	0.75	2.25	3.75	5.25	6.75	8.25	30.96	92.88	154.80	216.72	278.64	340.56
1′	138	4.22	3.635	14.39	16.71	70.52	0.25	0.75	1.25	1.75	2.25	2.75	4.18	12.54	20.90	29.26	37.62	45.98
						12926.24							2035.17	5904.87	9279.25	11957.99	13810.05	14754.62

Load of 1 lb. at Sect. 1: $H=\frac{2035.17}{12926.24}=0.1575$ Load of 1 lb. at Sect. 2: $H=\frac{5904.87}{12926.24}=0.4568$ Load of 1 lb. at Sect. 3: $H=\frac{9279.25}{12926.24}=0.7179$

Load of 1 lb. at Sect. 4: $H=\frac{11957.99}{12926.24}=0.9252$ Load of 1 lb. at Sect. 5: $H=\frac{13810.05}{12926.24}=1.0683$ Load of 1 lb. at Sect. 6: $H=\frac{14754.62}{12926.24}=1.1414$

TABLE No. 39b

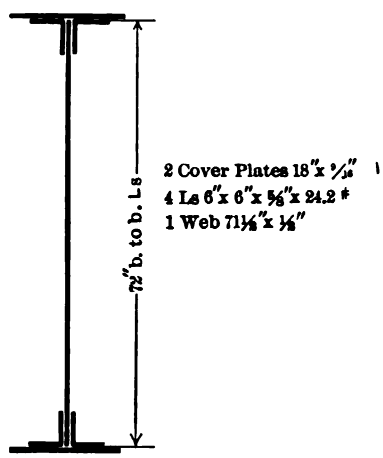
VALUES OF H AND V FOR LOAD OF 1 LB. AT THE VARIOUS SECTIONS

Load 1 lb. at	1	2	3	4	5	6	6′	5′	4′	3′	2′	1′	Σ
H	0.1575	0.4568	0.7179	0.9252	1.0683	1.1414	1.1414	1.0683	0.9252	0.7179	0.4568	0.1575	8.9342
V_L	0.9583	0.8750	0.7917	0.7084	0.6250	0.5417	0.4583	0.3750	0.2916	0.2083	0.1250	0.0417	6.0000
V_R	0.0417	0.1250	0.2083	0.2916	0.3750	0.4583	0.5417	0.6250	0.7084	0.7917	0.8750	0.9583	6.0000

For the case under consideration the depth of rib will be taken as 72 ins. out to out of flange angles.

$$A = \frac{1}{10,000}\left(331,200 + \frac{19,077,120}{36.5}\right) = \frac{331,200 + 522,700}{10,000} = 854 \text{ sq.ins.}$$

The rib for this investigation will be taken as shown in Fig. 39*d*. The computations should be carefully tabulated, just as was done for the arch with no hinges. The rib for this case being divided into twelve equal parts with respect to the span length.



SECTION

FIG. 39*d*.

The first table to be prepared is that of 39*a*, in which all the values of Δ and d required to determine the thrusts are computed. Just below this table are given the values for the thrust for a load at the center of each of the twelve divisions.

Table No. 39*b* may now be prepared.

With Table No. 39*b* the influence drawing of Fig. 39*e* may be prepared.

The lines within the rib which show whether any load produces tension or compression at any point on any cross-section are drawn in light, full lines according to the method established in Art. 36.

By means of the influence drawing, Table No. 39*c* is prepared. This table shows the loading which must be used for maximum stresses of either tension or compression on the twelve selected sections of the arch.

Values of the horizontal and vertical components and the moment of the thrust which produces the maximum tensile and compressive unit stress at each section, should now be computed for the loadings of the influence table No. 39*c*, and these horizontal and vertical components and moments should be recorded as in Table No. 39*d*. In the preparation of this table a temperature range of $\pm 60°$ F., and a modulus of elasticity of 29,000,000 for steel, were assumed.

Giving for this thrust

$$H_t = \frac{\Delta_t}{d} = \frac{60 \times 144 \times .0000065}{\dfrac{12,926.4}{29,000,000 \times 144}} = \frac{60 \times .0000065 \times 29,000,000 \times (144)^2}{12,926.4}$$

$$= \frac{.00039 \times 29,000,000 \times 20,736}{12,926.4} = 17,910 \text{ lbs., nearly.}$$

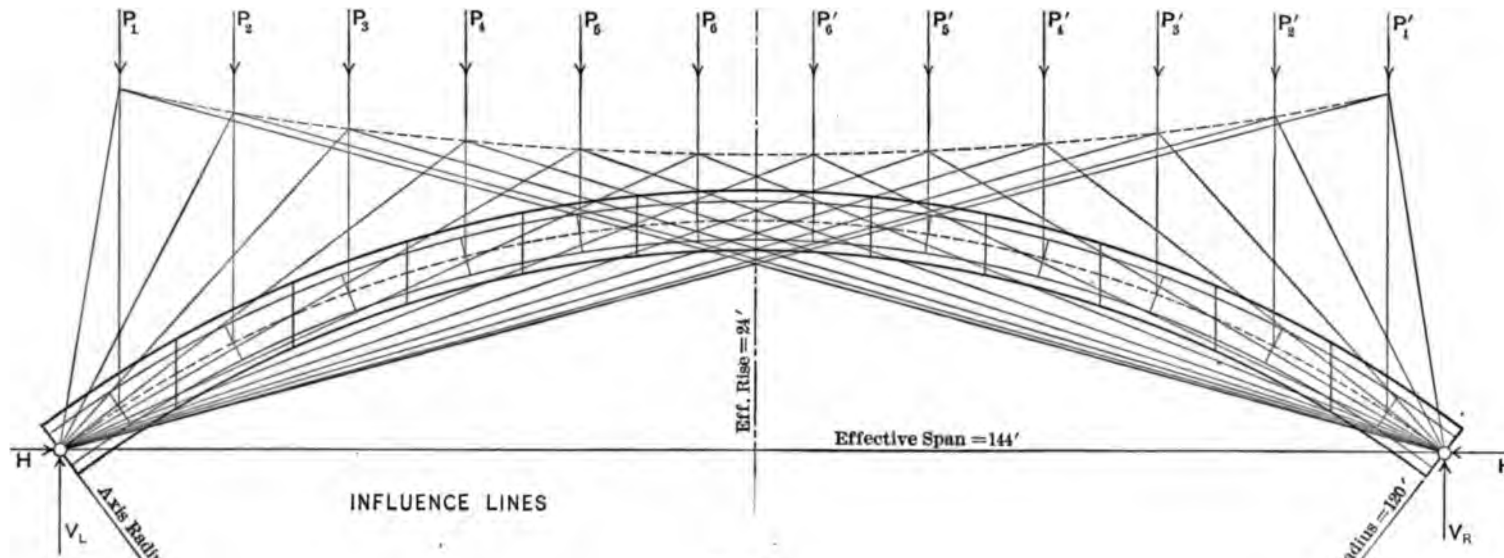


FIG. 39e.

TABLE NO. 39c—INFLUENCE TABLE FOR MAXIMUM COMPRESSION AND TENSION

		Div. No. 1	Div. No. 2	Div. No. 3	Div. No. 4	Div. No. 5	Div. No. 6
GROUP I Maximum compression on upper fiber	D.L.	All	All	All	All	All	All
	L.L.	P_1-P_5'	P_1-P_5	P_1-P_5	P_1-P_6	P_1-P_6'	P_2-P_5'
	Temp.	A fall	A fall	A fall	A fall	A fall	A fall
	Rib Sh.	All	All	All	All	All	All
GROUP II Maximum compression on lower fiber	D.L.	All	All	All	All	All	All
	L.L.	P_3-P_1'	P_5-P_1'	P_6-P_1'	P_6'-P_1'	P_6'-P_1'	P_1-P_4 & P_5'-P_1'
	Temp.	A rise	A rise	A rise	A rise	A rise	A rise
	Rib Sh.	All	All	All	All	All	All
GROUP III Max. tens. lower fiber	L.L.	P_1-P_2	P_1-P_4	P_1-P_4	P_1-P_5	P_1-P_6	P_5-P_6'
		D.L., Temp., and Rib Sh. same as for Group I					
GROUP IV Max. tens. upper fiber	L.L.	P_4'-P_1'	P_6-P_1'	P_6-P_1'	P_6'-P_1'	P_5'-P_1'	P_1' & P_4'-P_1'
		D.L., Temp., and Rib Sh. same as for Group II					

TABLE No. 39*d*

VALUES OF V, H, AND M AT EACH DIVISION FOR LOADINGS OF GROUPS I, II, III AND IV

Group.	Loading.	V_1	H_1	M_1	V_2	H_2	M_2	V_3	H_3	M_3
I	D.L.	+ 51600	− 76834	− 14600	+ 43000	− 76834	− 31900	+ 34400	− 76834	− 22400
	L.L.	+160000	−200304	+ 114600	+ 88752	− 99771	+ 664200	+ 58752	− 99771	+ 845100
	Temp.	0	+ 17910	+ 75600	0	+ 17910	+ 199900	0	+ 17910	+ 293900
	Rib Sh.	0	+ 6552	+ 27700	0	+ 6552	+ 73100	0	+ 6552	+ 107500
	Total	+211600	−252676	+ 203300	+131752	−152143	+ 905300	+ 93152	−152143	+1224100
II	D.L.	+ 51600	− 76834	− 14600	+ 43000	− 76834	− 31900	+ 34400	− 76834	− 22400
	L.L.	+125000	−249597	− 303300	+ 79998	−200304	− 795300	+ 79998	−200304	− 887100
	Temp.	0	− 17910	− 75600	0	− 17910	− 199900	0	− 17910	− 293900
	Rib Sh.	0	+ 6552	+ 27700	0	+ 6552	+ 73100	0	+ 6552	+ 107500
	Total	+176600	−337789	− 365800	+122998	−288496	− 954000	+114398	−288496	−1095900
III	D.L.	+ 51600	− 76834	− 14600	+ 43000	− 76834	− 31900	+ 34400	− 76834	− 22400
	L.L.	+ 55000	− 18430	+ 252300	+ 70000	− 67722	+ 684300	+ 40000	− 67722	+ 808500
	Temp.	0	+ 17910	+ 75600	0	+ 17910	+ 199900	0	+ 17910	+ 293900
	Rib Sh.	0	+ 6552	+ 27700	0	+ 6552	+ 73100	0	+ 6552	+ 107500
	Total	+106600	− 70802	+ 341000	+113000	−120094	+ 925400	+ 74400	−120094	+1187500
IV	D.L.	+ 51600	− 76834	− 14600	+ 43000	− 76834	− 31900	+ 34400	− 76834	− 22400
	L.L.	+ 20000	− 67722	− 165600	+ 61250	−168255	− 775200	+ 61250	−168255	− 923700
	Temp.	0	− 17910	− 75600	0	− 17910	− 199900	0	− 17910	− 293900
	Rib Sh.	0	+ 6552	+ 27700	0	+ 6552	+ 73100	0	+ 6552	+ 107500
	Total	+ 71600	−155914	− 228100	+104250	−256447	− 933900	+ 95650	−256447	−1132500

Group.	Loading.	V_4	H_4	M_4	V_5	H_5	M_5	V_6	H_6	M_6
I	D.L.	+ 25800	− 76834	− 3200	+ 17200	− 76834	+ 15000	+ 8600	− 76834	+ 25200
	L.L.	+ 45000	−134013	+ 804300	+ 28752	−168225	+ 622500	+ 11250	−195579	+ 397500
	Temp.	0	+ 17910	+ 361600	0	+ 17910	+ 405500	0	+ 17910	+ 427200
	Rib Sh.	0	+ 6552	+ 132300	0	+ 6552	+ 148300	0	+ 6552	+ 156300
	Total	+ 70800	−186385	+1295000	+ 45952	−220597	+1191300	+ 19850	−247951	+1006200
II	D.L.	+ 25800	− 76834	− 3200	+ 17200	− 76834	+ 15000	+ 8600	− 76834	+ 25200
	L.L.	+ 61251	−168225	− 824700	+ 45000	−134013	− 604500	+ 11250	−167493	− 372300
	Temp.	0	− 17910	− 361600	0	− 17910	− 405500	0	− 17910	− 427200
	Rib Sh.	0	+ 6552	+ 132300	0	+ 6552	+ 148300	0	+ 6552	+ 156300
	Total	+ 87051	−256417	−1057200	+ 62200	−222205	− 846700	+ 19850	−255685	− 618000
III	D.L.	+ 25800	− 76834	− 3200	+ 17200	− 76834	+ 15000	+ 8600	− 76834	+ 25200
	L.L.	+ 28750	− 99770	+ 813300	+ 15000	−134013	+ 555500	+ 18750	−100533	+ 459300
	Temp.	0	+ 17910	+ 361600	0	+ 17910	+ 405500	0	+ 17910	+ 427200
	Rib Sh.	0	+ 6552	+ 132300	0	+ 6552	+ 148300	0	+ 6552	+ 156300
	Total	+ 54550	−152142	+1304000	+ 32200	−186395	+1124300	+ 27350	−152905	+1068000
IV	D.L.	+ 25800	− 76834	− 3200	+ 17200	− 76834	+ 15000	+ 8600	− 76834	+ 25200
	L.L.	+ 45000	−134013	− 817500	+ 31248	− 99770	− 571500	+ 18750	− 72450	− 310500
	Temp.	0	− 17910	− 361600	0	− 17910	− 405500	0	− 17910	− 427200
	Rib Sh.	0	+ 6552	+ 132300	0	+ 6552	+ 148300	0	+ 6552	+ 156300
	Total	+ 70800	−222205	−1050000	+ 48448	−187962	− 813700	+ 27350	−160642	− 556200

The rib shortening is taken as equal to that due to the thrust for full live and dead load for a cross-section of 84.44 sq.ins. and 144 ft. long.

$$H_{rs}=\frac{\Delta_{rs}}{d}=\frac{344{,}860\times 144}{84.44\times 29{,}000{,}000}\div\frac{12{,}926.4}{29{,}000{,}000\times 144}=\frac{344{,}860\times 20{,}736}{84.44\times 12{,}926.4}$$

$$=6552 \text{ lbs., nearly.}$$

With Table No. 39*d* prepared the final table No. 39*e* is made, the last column of which gives the maximum unit stress and its character for each of the twelve cross-sections. For this case the maximum unit stresses are found to be in the neighborhood of 10,000 lbs. per sq.in., and therefore the sections do not need revision.

Table No. 39*e*

VALUES OF AND UNIT STRESSES DUE TO THRUST AND MOMENT FOR GROUPS I, II, III, AND IV

Group.	Sect.	$\sqrt{V^2+H^2}$	A sq.ins.	$\frac{\sqrt{V^2+H^2}}{A}$	M in.-lbs.	$\frac{I}{c}$(ins.)^3	$\frac{Mc}{I}$ lbs. per sq.in.	Maximum Compression and Tension.
I	1	329300	84.44	3900	+ 2439600	2061.7	+1183	+5083
$\frac{\sqrt{V^2+H^2}}{A}+\frac{Mc}{I}$	2	201300	84.44	2384	+10863600	2061.7	+5270	+7654
Maximum compression on upper fiber	3	178400	84.44	2113	+14689200	2061.7	+7126	+9239
	4	199400	84.44	2361	+15540000	2061.7	+7539	+9900
	5	225300	84.44	2668	+14295600	2061.7	+6935	+9603
	6	248700	84.44	2945	+12074400	2061.7	+5858	+8803
II	1	381160	84.44	4514	− 4389600	2061.7	−2129	+6643
$\frac{\sqrt{V^2+H^2}}{A}-\frac{Mc}{I}$	2	313600	84.44	3714	−11448000	2061.7	−5553	+9267
Maximum compression on lower fiber	3	310350	84.44	3676	−13150800	2061.7	−6380	+10056
	4	270800	84.44	3207	−12686400	2061.7	−6154	+9361
	5	230750	84.44	2733	−10160400	2061.7	−4928	+7661
	6	256450	84.44	3037	− 7416000	2061.7	−3597	+6634
III	1	127960	84.44	1515	+ 4092000	2061.7	+1985	− 470
$\frac{\sqrt{V^2+H^2}}{A}-\frac{Mc}{I}$	2	164900	84.44	1953	+11104800	2061.7	+5388	−3435
Maximum tension on lower fiber	3	141270	84.44	1673	+14250000	2061.7	+6912	−5239
	4	161630	84.44	1914	+15648000	2061.7	+7590	−5676
	5	189170	84.44	2240	+13491600	2061.7	+6544	−4304
	6	155330	84.44	1840	+12816000	2061.7	+6217	−4377
IV	1	171580	84.44	2032	− 2737200	2061.7	−1328	+ 704
$\frac{\sqrt{V^2+H^2}}{A}+\frac{Mc}{I}$	2	276830	84.44	3279	−11206800	2061.7	−5436	−2157
Maximum tension on upper fiber	3	273650	84.44	3241	−13590000	2061.7	−6592	−3351
	4	233220	84.44	2762	−12600000	2061.7	−6112	−3350
	5	194100	84.44	2299	− 9764400	2061.7	−4736	−2437
	6	182450	84.44	2161	− 6664400	2061.7	−3232	−1071

+ = Compression. − = Tension.

PROBLEMS

No. **39*a***. Correct the tables of this article for an assumed temperature range of ±75° F. and for a modulus of elasticity of 27,000,000.

No. **39*b***. Correct the tables of this article for a rib shortening produced by a thrust of 330,000 lbs. and for a modulus of elasticity of 27,000,000.

ART. 40. SHEARING STRESSES AT ANY CROSS-SECTION

In order to properly proportion the rivet spacing for connecting the flanges of the rib to the web and the component parts of the flanges together it will be necessary to know the maximum shearing stress at any cross-section. The influence drawing, Fig. 39*e*, gives all the information required to determine the loading necessary to produce either positive or negative shear. A negative shear will be considered to be that produced by a component of the thrust on the right of any section which acts toward the center of the arch axis. A positive shear will be taken as acting in a direction opposite to the negative shear. The resultant of all the forces to the right of any section, acting on any radial cross-section, gives a negative shear when it has a component which acts along the section and toward the center.

TABLE No. 40*a*

INFLUENCE TABLE FOR MAXIMUM SHEAR

		Div. No. 1.	Div. No. 2.	Div. No. 3.	Div. No. 4.	Div. No. 5.	Div. No. 6.
Group I	D.L.	All	All	All	All	All	All
Maximum	L.L.	P_5-P_1'	P_1 & P_6-P_1'	P_1-P_2&P_5'-P_1'	P_1-P_3	P_1-P_4	P_1-P_5
positive	Temp.	A rise	A rise	A rise	A rise	A rise	A rise
shear	Rib Sh.	All	All	All	All	All	All
Group II	D.L.	All	All	All	All	All	All
Maximum	L.L.	P_1-P_4	P_2-P_5	P_3-P_6'	P_4-P_1'	P_5-P_1'	P_6-P_1'
negative	Temp.	A fall	A fall	A fall	A fall	A fall	A fall
shear	Rib Sh.	All	All	All	All	All	All

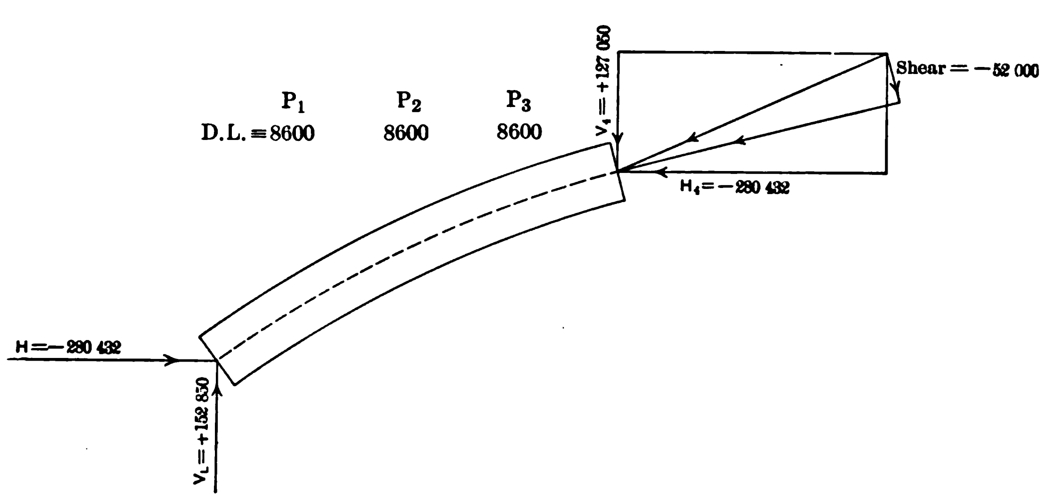


Fig. 40*a*.—Maximum Negative Shear at Div. No. 4.

From these considerations Table No. 40*a* is prepared and with this table the maximum positive and negative shears at each section may be computed if desired. For example, let the maximum negative shear at the center of Division No. 4 be required.

For dead load, from Table No. 39*d*,	$V_4 = +\ 25{,}800$	and	$H_4 = -\ 76{,}834$
For live load, from Table No. 39*b*,	$V_4 = +101{,}250$	and	$H_4 = -228{,}060$
$V_4 = 3.375 \times 30{,}000 = 101{,}250$			
$H_4 = 7.602 \times 30{,}000 = 228{,}060$			
For temperature, from Table No. 39*d*,	$V_4 = \quad 0$	and	$H_4 = +\ 17{,}910$
For rib shortening, from Table No. 39*d*,	$V_4 = \quad 0$	and	$H_4 = +\ 6{,}552$
Totals,	$V_4 = +127{,}050$	and	$H_4 = -280{,}432$

With these values of V_4 and H_4 the resultant thrust on the cross-section No. 4 may be determined graphically, as is shown in Fig. 40*a*. The component of this thrust acting along the cross-section toward the center of the ring axis is determined graphically to be 52,000 lbs.

PROBLEM

No. **40*a*.** Find the maximum positive shear on the cross-section of Div. No. 1, for the arch of Arts. 39 and 40.

CHAPTER V

DEFLECTIONS OF STRUCTURES WITH EITHER SOLID OR OPEN WEBS

ART. 41. THE FIRST THEOREM OF CASTIGLIANO

THE displacement of the point of application of an external force, in the direction of the force, acting on an elastic body subject to any loading is equal to the first differential coefficient of the total work of resistance with respect to the force whose displacement is desired.

Let Fig. 41*a* represent an elastic body subject to the forces or loads P_1, P_2, P_3, etc., and let Δ_1, Δ_2, Δ_3, etc., be the total deflections of the points of

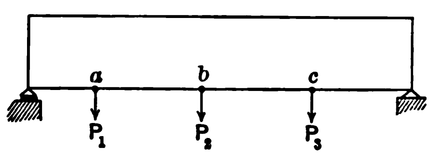


FIG. 41*a*.

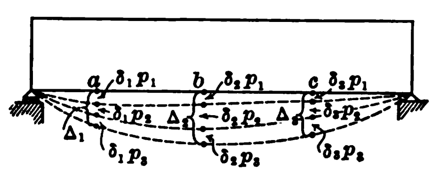


FIG. 41*b*.

application *a*, *b*, and *c* respectively of P_1, P_2, P_3, etc. as shown in Fig. 41*b*. Now

$$\Delta_1 = \delta_1 p_1 + \delta_1 p_2 + \delta_1 p_3 + \text{etc.},$$

$$\Delta_2 = \delta_2 p_1 + \delta_2 p_2 + \delta_2 p_3 + \text{etc.},$$

and

$$\Delta_3 = \delta_3 p_1 + \delta_3 p_2 + \delta_3 p_3 + \text{etc.},$$

in which $\delta_1 p_1$, $\delta_2 p_1$, and $\delta_3 p_1$ are the deflections of *a*, *b*, and *c* respectively due to P_1, $\delta_1 p_2$, $\delta_2 p_2$, and $\delta_3 p_2$, and $\delta_1 p_3$, $\delta_2 p_3$, and $\delta_3 p_3$ are the same as before defined for *a*, *b*, and *c* due to P_2 and P_3.

Let it be supposed that the loads P_1, P_2, and P_3 are gradually applied, that is, that they increase gradually from zero to their full amount, and let K represent the total work of the system produced by moving the loads through their respective deflections; then

$$K = \frac{P_1\Delta_1}{2} + \frac{P_2\Delta_2}{2} + \frac{P_3\Delta_3}{2} = \frac{P_1}{2}(\delta_1 p_1 + \delta_1 p_2 + \delta_1 p_3) + \frac{P_2}{2}(\delta_2 p_1 + \delta_2 p_2 + \delta_2 p_3)$$
$$+ \frac{P_3}{2}(\delta_3 p_1 + \delta_3 p_2 + \delta_3 p_3).$$

If it now be supposed that P_2 takes an increment dP_2, the deflections at each of the three loads is increased. Let dK be the increment in the total work due to this cause and let $\delta_1 p_2$, $\delta_2 p_2$, and $\delta_3 p_2$, the deflections at P_1, P_2, and P_3 due to P_2, be rewritten in terms of the load P_2, that is, as functions of this load as follows

$$\delta_1 p_2 = f_1(P_2) = \frac{P_2}{n},$$

$$\delta_2 p_2 = f_2(P_2) = \frac{P_2}{m},$$

and

$$\delta_3 p_2 = f_3(P_2) = \frac{P_2}{o}.$$

The increased work done at P_1

$$= P_1 \times \delta_1 p_2 \times \frac{dP_2}{P_2} = P_1 \times \frac{P_2}{n} \times \frac{dP_2}{P_2} = \frac{P_1}{n} \times dP_2,$$

and neglecting the higher powers of infinitesimals the increase at

$$P_2 = (P_2 + dP_2)\left(\delta_2 p_2 \times \frac{dP_2}{P_2}\right) = (P_2 + dP_2)\left(\frac{P_2}{m} \times \frac{dP_2}{P_2}\right) = P_2 \frac{dP_2}{m} + \frac{dP_2{}^2}{m} = \frac{P_2}{m} \times dP_2,$$

and at

$$P_3 = P_3 \times \delta_3 p_2 \times \frac{dP_2}{P_2} = P_3 \times \frac{P_2}{o} \times \frac{dP_2}{P_2} = \frac{P_3}{o} dP_2.$$

Then

$$dK = \frac{P_1}{n} dP_2 + \frac{P_2}{m} dP_2 \times \frac{P_3}{o} dP_2,$$

and

$$\frac{dK}{dP_2} = \frac{P_1}{n} + \frac{P_2}{m} + \frac{P_3}{o}. \quad \quad (1)$$

Rewriting the expression for work we have

$$2K = P_1\left(\delta_1 p_1 + \frac{P_2}{n} + \delta_1 p_3\right) + P_2\left(\delta_2 p_1 + \frac{P_2}{m} + \delta_2 p_3\right) + P_3\left(\delta_3 p_1 + \frac{P_2}{o} + \delta_3 p_3\right),$$

and differentiating with respect to P_2,

$$2dK = \frac{P_1 dP_2}{n} + \Delta_2 dP_2 + \frac{P_2 dP_2}{m} + \frac{P_3 dP_2}{o},$$

$$\frac{2dK}{dP_2} = \frac{P_1}{n} + \Delta_2 + \frac{P_2}{m} + \frac{P_3}{o}. \quad \quad (2)$$

Subtracting (1) from (2) it is seen that

$$\frac{dK}{dP_2} = \Delta_2.$$

In the same manner it can be shown that

$$\frac{dK}{dP_1}=\Delta_1, \quad \text{and} \quad \frac{dK}{dP_3}=\Delta_3,$$

and the demonstration can be similarly extended for any number of loads.

No restriction has been placed on either the amount or direction of the given loading or the form or character of the elastic body, so the theorem is fully demonstrated.

In making practical application of this method to finding deflections it is necessary to write the expression for the work of the system of external loads as the sum of the work done in deforming each bar or element of the structure, as follows:

Let S be the stress in any bar due to a given loading, the bar having a length L and a cross-section A.

The change in length of the bar being $\frac{SL}{EA}$, and as S is gradually applied, if the loads which produce it are, the average stress is $\frac{S}{2}$, and the work in this bar $\frac{S}{2}\cdot\frac{SL}{EA}=\frac{S^2L}{2EA}$.

The stress S may be considered as made up of parts each of which is a stress produced by one of the given loads. As the deflection of a certain point in a definite direction is desired for a given system of loads, an auxiliary load acting at the point and in the direction of which the deflection is desired will be employed. Calling this auxiliary load D and the stress it produces in any bar $f(D)$ the expression for the work of resistance in this bar $=\frac{[S+f(D)]^2L}{2EA}$, and the total interval work of the structure $K=\Sigma\frac{[S+f(D)]^2L}{2EA}$. This expression being true for all values of D is true for the value when $D=0$.

Hence if we find $\frac{dK}{dD}$ for all the bars in terms of S and D and find its value when $D=0$, we will have the desired deflection, that is, the deflection for the given loading, as the auxiliary load has been made equal to zero.

As an example of the application of this method: Take the truss of Fig. 41*c*, and let it be required to find the deflection at the center bottom chord point. The dead load is 9500 lbs. per panel—1900 lbs. at the top and 7600 lbs. at the bottom. The live load is 15,200 per panel, all on the bottom.

The computations may be conveniently arranged, as shown in Table No. 41*a*. This table shows so clearly the successive steps that no comment is necessary.

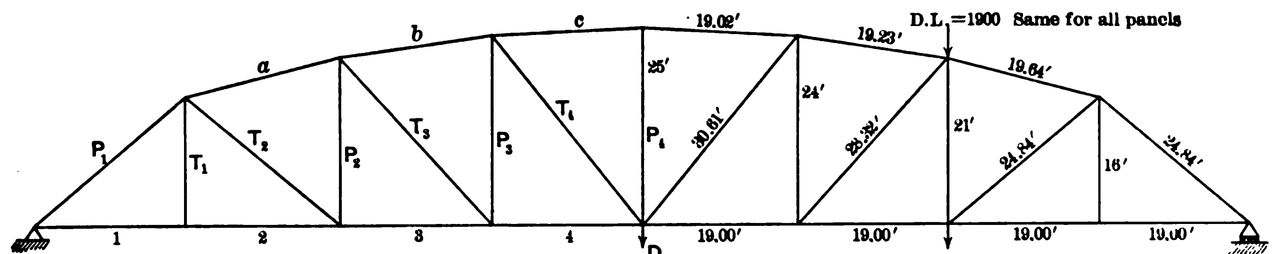


Fig. 41c.

Table No. 41a

Member.	Length, L	Area of Cross-section, A	$\frac{L}{A}$	Live+Dead Stress $=S=$ $DL+LL$	Stress Due to Load D	Work Done in Each Member, K_p.	$\frac{dK_p}{dD}$	$\frac{dK_p}{dD}/D=0$
P_1	24.84	16.72	1.49	-134100	$-.776D$	$\frac{1.49}{2E}[-134100-.776D]^2$	$\frac{1.49}{E}[134100+.776D]\ .776$	$+154.80\times\frac{1000}{E}$
P_2	21.00	5.29	3.97	-3700	$-.262D$	$\frac{3.97}{2E}[-3700-.262D]^2$	$\frac{3.97}{E}[3700+.262D]\ .262$	$+3.80\times$ "
P_3	24.00	5.29	4.54	$+8900$	$-.312D$	$\frac{4.54}{2E}[8900-.312D]^2$	$\frac{4.54}{E}[8900-.312D]-.312$	$-12.62\times$ "
P_4	25.00	5.29	4.72	$+13900$	$+.160D$	$\frac{4.72}{2E}[13900+.160D]^2$	$\frac{4.72}{E}[13900+.160D]\ .160\div 2$	$+5.25\times$ "
T_1	16.00	5.29	3.03	$+22800$	$.000D$	$\frac{3.03}{2E}[22800\quad]^2$	$\frac{3.03}{E}[\ 0\ ]$	
T_2	24.84	3.75	6.62	$+41000$	$+.407D$	$\frac{6.62}{2E}[41000+.407D]^2$	$\frac{6.62}{E}[41000+.407D]\ .407$	$+110.40\times$ "
T_3	28.32	7.06	4.01	$+18700$	$+.420D$	$\frac{4.01}{2E}[18700+.420D]^2$	$\frac{4.01}{E}[18700+.420D]\ .420$	$+31.49\times$ "
T_4	30.61	7.06	4.33	$+5800$	$+.536D$	$\frac{4.33}{2E}[5800+.536D]^2$	$\frac{4.33}{E}[5800+.536D]\ .536$	$+13.46\times$ "
a	19.64	13.89	1.41	-139000	$-.935D$	$\frac{1.41}{2E}[-139000-.935D]^2$	$\frac{1.41}{E}[139000+.935D]\ .935$	$+183.20\times$ "
b	19.23	13.89	1.38	-148200	$-1.199D$	$\frac{1.38}{2E}[-148200-1.199D]^2$	$\frac{1.38}{E}[148200+1.199D]1.199$	$+244.90\times$ "
c	19.02	13.89	1.37	-150300	$-1.522D$	$\frac{1.37}{2E}[-150300-1.522D]^2$	$\frac{1.37}{E}[150300+1.522D]1.522$	$+313.40\times$ "
1	19.00	4.50	4.22	$+102600$	$+.593D$	$\frac{4.22}{2E}[102600+.593D]^2$	$\frac{4.22}{E}[102600+.593D]\ .593$	$+256.70\times$ "
2	19.00	4.50	4.22	$+102600$	$+.593D$	$\frac{4.22}{2E}[102600+.593D]^2$	$\frac{4.22}{E}[102600+.593D]\ .593$	$+256.70\times$ "
3	19.00	8.75	2.17	$+134000$	$+.905D$	$\frac{2.17}{2E}[134000+.905D]^2$	$\frac{2.17}{E}[134000+.905D]\ .905$	$+263.10\times$ "
4	19.00	9.37	2.03	$+146600$	$+1.187D$	$\frac{2.03}{2E}[146600+1.187D]^2$	$\frac{2.03}{E}[146600+1.187D]1.187$	$+353.20\times$ "

$$\Sigma\frac{dK_p}{dD}\times 2=\Delta \qquad 2177.78\times\frac{1000}{E}$$

$$\Delta=(E=30000000)=.1451\text{ feet}$$

PROBLEM

41a. Find the horizontal deflection of the right-hand end bottom chord point of the truss of Fig. 41c by the method of this article.

ART. 42. THE SECOND THEOREM OF CASTIGLIANO

The differential coefficient of the total work of resistance due to any given loading with respect to a particular force so chosen that the force does no work is equal to zero.

It has been shown that $\frac{dK}{dP} = \Delta$ in the preceding article and that the load P may be in any direction at any point of the structure and of any amount, as no restriction whatever was placed on the general nature of P. Therefore the above theorem is a corollary of the first theorem. The following examples will illustrate it and make its meaning and use clear.

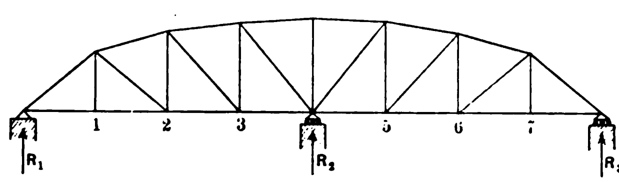


FIG. 42a.

Let Fig. 42a, which is the same structure subject to the same loads as shown previously in Fig. 41c, and again at the head of the table in Fig. 42b, be supported at the center, and let it be required to determine the reaction R_2 when the supports are on the same level. This is simply a problem to determine the amount of R_2 necessary to have zero vertical deflection at the center.

Columns No. 6 and 7 of the following table, No. 42a, give the stresses due to an upward load R_2 and the value of $\frac{dK}{dR_2}$ for each member. The remainder of the table will be the same as for the table of the preceding article. Taking the sum of $\frac{dK}{dR_2}$ for all the members of the truss and writing it equal to zero, and solving for R_2 we obtain such a value of R_2 as must exist when the structure rests on three supports.

This problem will be solved in a later part of the book by other methods so that comparison of methods will be possible.

PROBLEMS

No. 42a. What are the reactions of the truss of Fig. 42a, when subject to a load of 100,000 lbs. at (1), the truss being free to move horizontally at R_1 and R_3 but held against all vertical motion?

No. 42b. What are the reactions of the truss of Fig. 42a, when subject to a load of 100,000 lbs. at (2)?

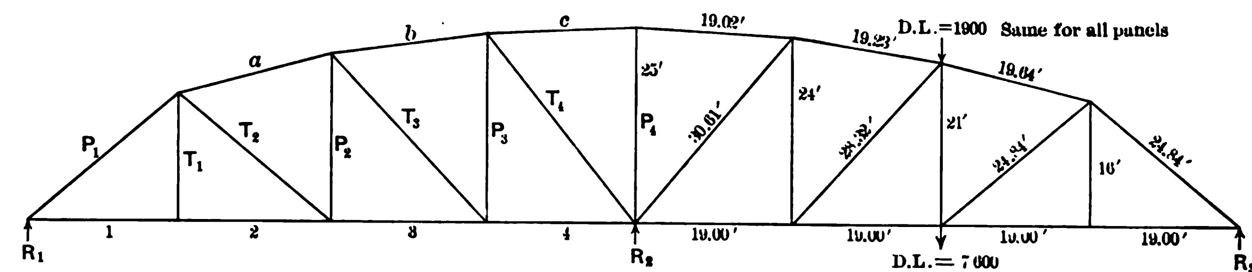


FIG. 42b.

TABLE No. 42a

Member	Length. L	Area of Cross-section. A	$\frac{L}{A}$	Live + Dead Stress. $DL+LL$	Stress Due to Load "R_2"	Work Done in Each Member.	$\frac{dK_p}{dR_2}$	$\frac{dK_p}{dR_2}\times E$
P_1	24.84	16.72	1.49	−134100	+ .776R_2	$\frac{1.49}{2E}[-134100+.776R_2]^2$	$\frac{1.49}{E}[-134100+.776R_2]$ .776	$[-154800+.897R_2]$
P_2	21.00	5.29	3.97	− 3700	+ .262R_2	$\frac{3.97}{2E}[-3700+.262R_2]^2$	$\frac{3.97}{E}[-3700+.262R_2]$ .262	$[-3800+.273R_2]$
P_3	24.00	5.29	4.54	+ 8900	+ .312R_2	$\frac{4.54}{2E}[8900+.312R_2]^2$	$\frac{4.54}{E}[+8900+.312R_2]$ .312	$[+12620+.442R_2]$
P_4	25.00	5.29	4.72	+ 13900	− .160R_2	$\frac{4.72}{2E}[13900-.160R_2]^2$	$\frac{4.72}{*2E}[-13900+.160R_2]$ .160	$[-5250+.060R_2]$
T_1	16.00	5.29	3.03	+ 22800	000	$\frac{3.03}{2E}[22800]^2$	$\frac{3.03}{E}[0]$	$[0]$
T_2	24.84	3.75	6.62	+ 41000	− .407R_2	$\frac{6.62}{2E}[41000-.407R_2]^2$	$\frac{6.62}{E}[-41000+.407R_2]$ .407	$[-110400+1.097R_2]$
T_3	28.32	7.06	4.01	+ 18700	− .420R_2	$\frac{4.01}{2E}[18700-.420R_2]^2$	$\frac{4.01}{E}[-18700+.420R_2]$ .420	$[-31490+.707R_2]$
T_4	30.61	7.06	4.33	+ 5800	− .536R_2	$\frac{4.33}{2E}[5800-.536R_2]^2$	$\frac{4.33}{E}[-5800+.536R_2]$ .536	$[-13460+1.243R_2]$
a	19.64	13.89	1.41	−139000	+ .935R_2	$\frac{1.41}{2E}[-139000+.935R_2]^2$	$\frac{1.41}{E}[-139000+.935R_2]$ .935	$[-183200+1.233R_2]$
b	19.23	13.89	1.38	−148200	+1.199R_2	$\frac{1.38}{2E}[-148200+1.199R_2]^2$	$\frac{1.38}{E}[-148200+1.199R_2]$1.199	$[-244900+1.984R_2]$
c	19.02	13.89	1.37	−150300	+1.522R_2	$\frac{1.37}{2E}[-150300+1.522R_2]^2$	$\frac{1.37}{E}[-150300+1.522R_2]$1.522	$[-313400+3.172R_2]$
1	19.00	4.50	4.22	+102600	− .593R_2	$\frac{4.22}{2E}[102600-.593R_2]^2$	$\frac{4.22}{E}[-102600+593R_2]$ .593	$[-256700+1.484R_2]$
2	19.00	4.50	4.22	+102600	− .593R_2	$\frac{4.22}{2E}[102600-.593R_2]^2$	$\frac{4.22}{E}[-102600+593R_2]$ .593	$[-256700+1.484R_2]$
3	19.00	8.75	2.17	+134000	− .905R_2	$\frac{2.17}{2E}[134000-.905R_2]^2$	$\frac{2.17}{E}[-134000+.905R_2]$ .905	$[-263100+1.777R_2]$
4	19.00	9.37	2.03	+146600	−1.187R_2	$\frac{2.03}{2E}[146600-1.187R_2]^2$	$\frac{2.03}{E}[-146600+1.187R_2]$1.187	$[-353200+2.861R_2]$

*Only ½ of this member is taken for ½ the Truss

$$\Sigma\frac{dK}{dR_2}=0=[-2177780+18.714R_2]$$

$$R=\frac{2\times 2177780}{2\times 18.714}=116200$$

No. **42c**. What are the reactions of the truss of Fig. 42*a*, when subject to a load of 100,000 lbs. at (3)?

No. **42d**. What are the reactions for Problems Nos. 42*a*, 42*b*, and 42*c* computed by means of the equation of the elastic curve if the truss be considered to have a constant moment of inertia.

ART. 43. MAXWELL'S RECIPROCAL THEOREM

The displacement of any point x of an elastic structure in a given direction, due to a force at any other point y of the structure, is equal to the displacement of the second point y in the direction of the force at that point due to a force of the same amount in the given direction at the first point x.

To illustrate, let Fig. 43*a* and Fig. 43*b* represent the same elastic structure. The load P_y at point y produces a displacement of point x in the direction of P_x of $\delta_x p_y$, which is equal to the displacement of point y, in the direction of P_y of $\delta_y p_x$ produced by P_x at x, provided P_x and P_y are equal forces.

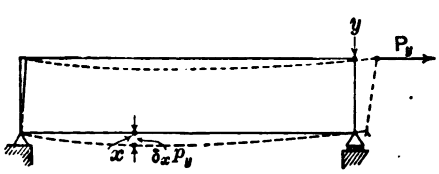


FIG 43*a*.

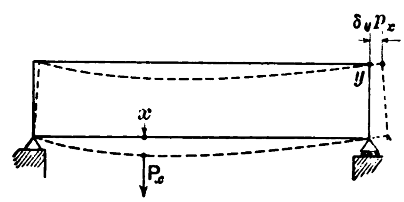


FIG. 43*b*.

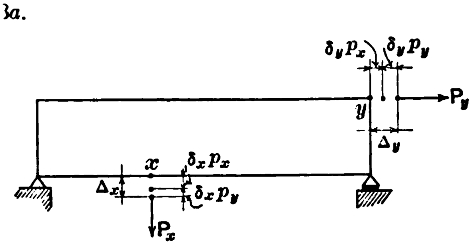


FIG. 43*c*.

To prove this, let Fig. 43*c* show the elastic structure of Figs. 43*a* and 43*b* loaded at the points x and y by P_x and P_y respectively. In Figs. 43*a*, 43*b*, and 43*c* no attempt has been made to show the deflections which would actually occur.

Let Δ_x be the total deflection of point x in the direction of P_x;

Δ_y be the total deflection of point y in the direction of P_y;

$\delta_y p_x = \frac{1}{l} . P_x =$ the deflection of y in the direction of P_y, due to P_x;

$\delta_y p_y = \frac{1}{m} . P_y =$ the deflection of y due to P_y in the direction of P_y;

$\delta_x p_x = \frac{1}{n} . P_x =$ the deflection of x due to P_x in the direction of P_x;

$\delta_x p_y = \frac{1}{o} . P_y$ = the deflection of x due to P_y in the direction of P_x;

K = total work of the system for the forces gradually applied.

Then

$$K = \frac{P_x}{2} . \Delta_x + \frac{P_y}{2} . \Delta_y = \frac{P_x}{2}\left(\frac{P_x}{n} + \frac{P_y}{o}\right) + \frac{P_y}{2}\left(\frac{P_x}{l} + \frac{P_y}{m}\right).$$

From Castigliano's First Theorem

$$\frac{dK}{dP_y} = \Delta_y = \frac{P_x}{2o} + \frac{P_x}{2l} + \frac{2P_y}{2m},$$

and when $P_y = 0$, this becomes

$$\frac{dK}{dP_y} = \delta_y p_x = \frac{P_x}{2o} + \frac{P_x}{2l}. \qquad (1)$$

Also

$$\frac{dK}{dP_x} = \Delta_x = \frac{2P_x}{2m} + \frac{P_y}{2o} + \frac{P_y}{2l},$$

and when $P_x = 0$

$$\frac{dK}{dP_x} = \delta_x p_y = \frac{P_y}{2o} + \frac{P_y}{2l}. \qquad (2)$$

Now if P_x and P_y are equal forces, the parts of Eq. (1) and (2) to the right of the equality sign are identical.

$$\therefore \quad \delta_y p_x = \delta_x p_y,$$

which proves the theorem as stated.

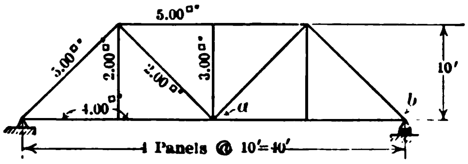


FIG. 43*d*.

PROBLEM

No. **43*a***. By means of Castigliano's First Theorem for the Truss of Fig. 43*d*: Find the horizontal motion of *b* due to a vertical downward force of 25,000 lbs. at *a* and find the vertical motion of *a* due to a horizontal force of 25,000 lbs. acting to the right at *b* and thereby show for this special case that Maxwell's Theorem is true.

CHAPTER VI

DEFLECTIONS AND STRESSES IN STRUCTURES WITH OPEN WEBS

ART. 44. DEFLECTIONS OF OPEN FRAMEWORKS BY THE METHOD OF WORK

LET Fig. 44*a* represent any statically determined structure. Let D be any force applied to the structure in any desired direction.

Let Δ_1 = deflection of the point to which D is applied and in the direction of action of D. Then

$$\text{the external work due to } D = \frac{D}{2} \cdot \Delta_1 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

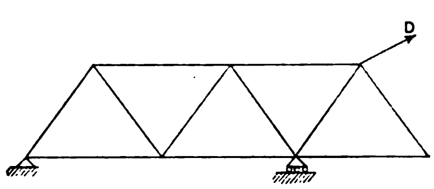


FIG. 44*a*.

Let T be the stress in any member of the truss due to D. Then

$$\text{the change in length of this member is } \frac{TL}{EA},$$

and

$$\text{the internal work in the member} = \frac{T}{2} \times \frac{TL}{EA} = \frac{T^2L}{2EA},$$

and

$$\text{the internal work for all the members} = \Sigma \frac{T^2L}{2EA} . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As the external and internal work must be equal, we have

$$\frac{D}{2} \cdot \Delta_1 = \Sigma \frac{T^2L}{2EA},$$

and

$$\Delta_1 = \Sigma \frac{T^2L}{DEA},$$

As it is generally required that the deflection of some particular point of a given structure in a given direction due to its own weight and some definite loading be known, it is not enough to be able to compute the deflection for a given point due to a load at that point only.

The following method will apply to the most general case.

Let the truss in Fig. 44*b* represent any truss under the loads a_1, a_2, a_3, a_4, a_5, etc.

Let it be required to find the motion of any point of the truss in any direction due to the loads a_1, a_2, a_3, a_4, a_5, etc.

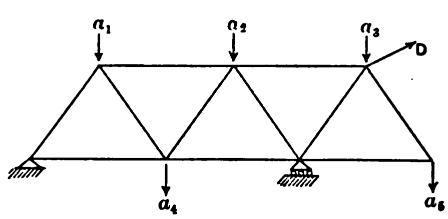


FIG. 44*b*.

Let a force D be applied to any point and acting in the desired direction.

Let S be stress in any member of the truss due to the loading for which its deflection is required. (The loads a_1, a_2, a_3, a_4, and a_5 of Fig. 44*b*.)

T be the stress in the same member due to D;

Δ = motion of the point whose deflection is required and in the desired direction due to loads a_1, a_2, a_3, a_4, a_5, etc.;

Δ_1 = motion of the point in the desired direction due to D;

Total deflection (in the desired direction) $= \Delta + \Delta_1$.

Total external work due to D gradually applied $= \frac{D}{2}(\Delta + \Delta_1)$. . . (1)

λ = elongation of any member due to S;

λ_1 = elongation of any member due to T. Now

$$\lambda = \frac{SL}{EA},$$

and

$$\lambda_1 = \frac{TL}{EA}.$$

The total internal work in any member due to D

$$=\frac{T}{2}(\lambda+\lambda_1)=\frac{T}{2}\left(\frac{SL}{EA}+\frac{TL}{EA}\right)=\frac{STL}{2EA}+\frac{T^2L}{2EA},$$

and

$$\text{the internal work for all the members due to } D=\Sigma\frac{STL}{2EA}+\Sigma\frac{T^2L}{2EA}. \quad . \quad (2)$$

Now (1) and (2) must be equal, whence

$$\frac{D}{2}(\Delta+\Delta_1)=\Sigma\frac{STL}{2EA}+\Sigma\frac{T^2L}{2EA},$$

$$\therefore \quad \Delta+\Delta_1=\Sigma\frac{STL}{DEA}+\Sigma\frac{T^2L}{DEA},$$

but

$$\Delta_1=\Sigma\frac{T^2L}{DEA},$$

as was shown in the first part of this article.

$$\therefore \quad \Delta=\Sigma\frac{STL}{DEA},$$

which is the formula for the desired deflection.

The value of D should always be taken as unity, and when so taken

$$\Delta=\Sigma\frac{SL}{EA}\times T.$$

Now $\frac{SL}{EA}$ = change in length of the member due to the given loads, or from any other cause, and therefore if λ, the total change in length for any member, be that due to the elastic action combined with any accidental or desired change of length, then $\Delta=\Sigma\lambda.T$ will be the desired deflection from all causes.

The arrangement of the computations necessary to determine the deflection at any desired point of a truss by the method of this article, will be shown in connection with the computation of the elastic deflection of the truss of Fig. 44c, which is the truss of Fig. 41c, and thereby we will be able to compare this method with that by means of Castigliano's First Theorem.

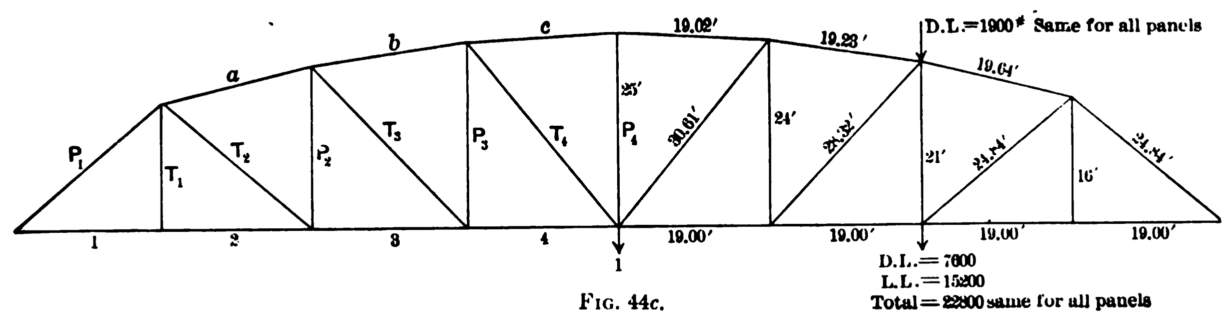


FIG. 44c.

TABLE No. 44c

	Length, L.	Area, A.	$\frac{L}{A}$	Dead + Live Stress, $DL + LL$	Elongations, $\frac{LS}{AE}$	Stress Due to Unity.	Deflection Increments.
P_1	24.84	16.72	1.49	−134100	−.0066210	− .776	+.005138
P_2	21.00	5.29	3.97	− 3700	−.0004896	− .262	+.000127
P_3	24.00	5.29	4.54	+ 8900	+.0013470	− .312	−.000420
P_4	25.00	5.29	4.72	+ 13900	+.0021870	+ .160 : 2 =	+.000175
T_1	16.00	5.29	3.03	+ 22800	+.0023028	.0	.000000
T_2	24.84	3.75	6.62	+ 41000	+.0090200	+ .407	+.003671
T_3	28.32	7.06	4.01	+ 18700	+.0024990	+ .420	+.001050
T_4	30.61	7.06	4.33	+ 5800	+.0008380	+ .536	+.000449
a	19.64	13.89	1.41	−139000	−.0065333	− .935	+.006109
b	19.23	13.89	1.38	−148200	−.0068162	−1.199	+.008173
c	19.02	13.89	1.37	−150300	−.0068162	−1.522	+.010374
1	19.00	4.50	4.22	+102600	+.0144324	+ .593	+.008558
2	19.00	4.50	4.22	+102600	+.0144324	+ .593	+.008558
3	19.00	8.75	2.17	+134000	+.0096832	+ .905	+.008763
4	19.00	9.37	2.03	+146600	+.0099197	+1.187	+.011775

$\Sigma = +.072500$
Sum $\times 2$ = def. = .1450 ft.

ART. 45. THE EFFECT OF PLAY OF PIN HOLES AND ERRORS IN PRODUCING DEFLECTIONS

The arrangement of the computations when changes of length other than that due to the elasticity of the material are to be included will be illustrated by the following simple case.

Let Fig. 45*a* show the outline and loading of a simple pin-connected Warren truss. The pin holes will be assumed to be $\frac{1}{32}$ in. larger than the pins.

The tension members should have their length center to center of pin holes increased by $\frac{1}{32}$ in., as is shown by Fig. 45*b*. Compression members should have their lengths decreased by $\frac{1}{32}$ in. Let it be required to find the horizontal motion of the right-hand hip.

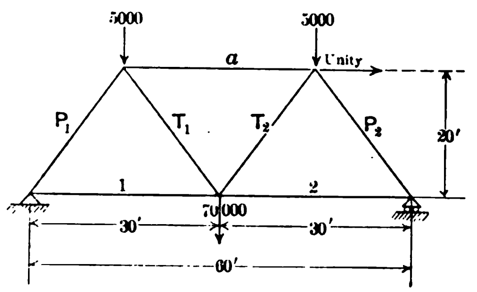


FIG. 45*a*.

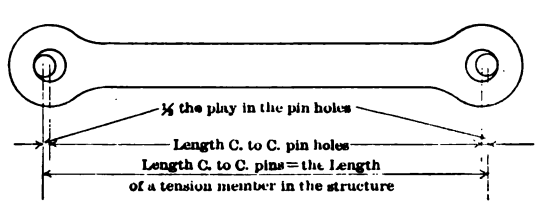


FIG. 45*b*.

The work of computing deflections where arbitrary changes in length of individual members are to be included, may be conveniently tabulated as shown in Table 45*a*, as follows:

TABLE No. 45*a*

Member.	Stress Due to Loading.	Areas in Sq.ins.	Lengths in Inches.	$\lambda_1 = \frac{SL}{EA}$ Inches.	λ_2 = Errors, Play of Pin Holes, etc.	Total $\lambda = \lambda_1 + \lambda_2$.	Stresses Due to Unity = T.	Partial Deflection $= \lambda \times T$.
a	−56250	6.0	360	−.1125	−.0313	−.1438	+.500	−.0719
1	+30000	3.0	360	+.1200	+.0313	+.1513	+.750	+.1135
2	+30000	3.0	360	+.1200	+.0313	+.1513	+.250	+.0378
P_1	−50000	6.0	300	−.0833	−.0313	−.1146	+.417	−.0478
T_1	+43750	4.0	300	+.1094	+.0313	+.1407	−.417	−.0587
P_2	−50000	6.0	300	−.0833	−.0313	−.1146	−.417	+.0478
T_2	+43750	4.0	300	+.1094	+.0313	+.1407	+.417	+.0587

Total deflection $= \Sigma\lambda T = (+.2578 - .1784) = +.0794''$

That is, the right-hand hip moves to the right .0794 in.

A study of the effect of each member in producing deflection is very instructive. A table such as the preceding will enable us to give any desired motion to any point of the structure.

For example, let it be required to alter the length of member (1) by an amount λ_c so that there will be no horizontal motion of the upper right-hand hip.

The length of this member before correction under the stated loads with $\frac{1}{32}$ in. play in the pin holes will be $360+.12+.0313=360.1513$ ins. c. to c. end pins.

The total horizontal motion of the point in question $=.0794$ in. to the right, or plus, as it is in the direction of the assumed force of unity.

The motion of this point due to a change λ_c in the length of (1) $=.750\times\lambda$.

Therefore for zero motion of the point $.750\times\lambda_c$ must equal .0794 in. and be of opposite $(-)$ sign, or $\lambda_c=\dfrac{-.0794}{.750}=-.1059$ in., and the member (1) should be manufactured $360.0000-.1059$ in. $=359.8941$ in. The length of (1) in the structure and under the given loading would be $359.8941+.1200+.0313=360.0450$ ins., giving for λ a value of .0454. If this value of λ be multiplied by T for (1) the partial deflection for (1) becomes .0361 instead of .1135 as tabulated, and $\Sigma\lambda T=0$.

In giving trusses a camber it is usual to correct the lengths of the various members for their elastic change and play of pin holes, but as the members are made to a duodecimal scale it is not possible to exactly correct them. Therefore to take care of the accumulation of such small errors one or two members can be corrected, as can be readily seen. The correction for the practical case should be so made that the resulting figure is symmetrical to as great a degree as it is possible to make it.

That is, if it is desirable to have no motion to the right of the right hand hip of the truss of this article, the correction should be made to both bottom chord members.

Changes in the lengths of members to give a desired deflection should be made in members which are most influential in producing such deflection.

PROBLEMS

No. **45a**. For the truss of this article, what is the vertical motion at the right-hand hip?

No. **45b**. What is the horizontal motion of the right-hand end bottom chord pin?

No. **45c**. How much motion, due to temperature changes, elastic deformation, and play of pin holes, must be provided for in the right-hand end bearing?

ART. 46. GRAPHICAL METHOD OF FINDING THE DISPLACEMENTS OF THE PANEL POINTS OF A SYMMETRICAL STRUCTURE, SYMMETRICALLY DEFORMED

The displacements which the panel points of a truss undergo due to a change in length of any or all of its members may readily be determined graphically, provided, as for all methods other than by geometry, the changes in length of the various members are not large enough to make the resulting truss materially different from the original figure. The method of geometry may be used to determine the location of the panel points of a framed structure no matter how great the resulting figure may vary from the original. For a truss figure which is symmetrical about its vertical center line both with respect to form of figure and area of cross-section of members, and at the same time symmetrically loaded or deformed, there will generally be at least one member which in the distorted figure is parallel to its position in the original figure.

Considering the displacements of the truss to take place with reference to some point on the member, or one of the members if there are more than one, which remains parallel to its original position, the displacements of all other points in the truss relative to this point are easily determined.

Let (*a*) of Fig. 46 be the two panels to the left of the center of the truss of Fig. 44*c* of page 140. The elastic change in length of the members of this truss are given in the sixth column of the table of page 140. These changes in length for the various members are so small that in order that they may be plotted to the scale to which (*a*) of Fig. 46 is drawn, they are multiplied by 100. Combining these elastic changes with the normal lengths of the members gives their length when the truss is under load. These lengths under load are given in feet on (*a*) of Fig. 46. The center post of this truss remains vertical after the truss is deflected by its loads. Let the point o be considered to undergo no displacement. Then w' is the new position of w; x', the new position of x, is now found by using the new lengths of ox and wx as indicated. From o and x' with the new lengths of op and xp the position of p', which is the new position of p, is found. In this manner the new positions of the remaining panel points are found with reference to o. As the original location of the panel points are correctly shown by the full line figure of (*a*) with reference to o, the new location of the panel points is their position with reference to the original, o being considered stationary and ow' remaining vertical. The new positions of the remaining panel points of the truss can be determined in the same manner.

The preceding method lacks accuracy, as the resulting figure of the truss, for the new lengths which are obtained by using changes 100 times too great, is not even for practical purposes similar to the actual deflected figure. It

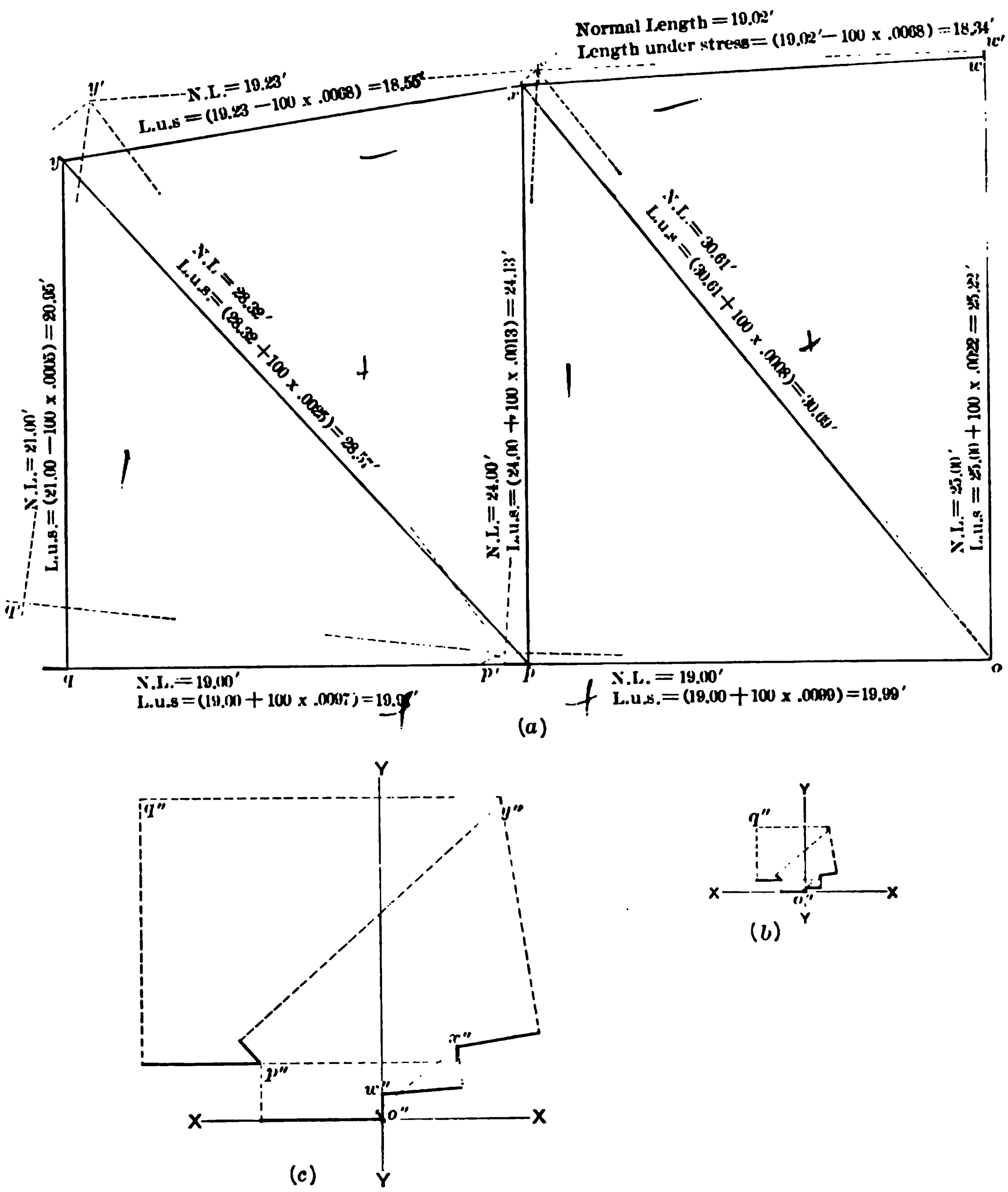


FIG. 46.

will be noted, however, that the new location of the panel points could be very accurately determined by using the true lengths of the members, when distorted, at a sufficiently large scale, and drawing lines perpendicular to the original

direction of each of the members at the ends which determine a panel point until they intersect, thus obtaining the new point. That is, to determine x' the length of ox is prolonged .0008 ft. and the new length laid off from o along the original direction of ox; from this new end a perpendicular is drawn extending upward and to the right, the length xw is shortened .0068 ft. and laid off from w' parallel to its original direction, and from the new end thus found a perpendicular extending upward is drawn, thus intersecting the first perpendicular in the new point x'. It may be further noted that if the length of the members be eliminated entirely from the graphical construction and only the changes in length of the various members used, then the resulting figure will have exactly the same degree of theoretical accuracy of the previous analytical methods.

In (*b*) and (*c*) of Fig. 46, let the point o'' be the origin of coordinates for the axes XX and YY. In (*b*) the changes in length are taken the same as in (*a*), while in (*c*) the changes in length are plotted five times as large as in (*a*) and (*b*) in order to show in a more distinct manner the construction of the figure.

Proceeding exactly as was done for (*a*), only omitting the lengths of the members entirely and using only the change in length of each member, the displacement diagrams, as they are called, (*b*) and (*c*), are constructed. The first few steps in the construction of these diagrams will be described in detail. As the member ow of (*a*) is lengthened but remains vertical the point w moves upward exactly the amount of the change in length of ow. Therefore the .0022 ft. laid off vertically from o'' the origin determines the point w''. A line joining o'' and w'' shows the displacement in direction and amount of w of the distorted figure with reference to w of the normal figure.

Having o'' and w'' the displaced positions of o and w we will find x'' the displaced position of x.

From w'' the change in length of $wx = .0068$ ft. is laid off to the right in the direction of wx as the point x moves to the right with reference to w as the member is shortened. From o'' the change in length of $ox = .0008$ ft. is laid off upward and to the left in the direction of ox as ox is lengthened.

The point x' of (*a*) is determined by describing arcs from o and w' with radii of ox' and $w'x'$ respectively. These radii move through extremely small arcs for the problems of practice. Therefore x'' may be determined in (*b*) and (*c*) by drawing perpendiculars to the changes in length of wx and ox at the ends which determine the new point and producing them till they intersect at x''.

A line joining o'' and x'' gives the displacement of x with reference to its former position.

Having the displaced position of x which is x'', and o which is o'', the displacement $o''p''$ of p may be found.

Having the displaced positions of x and p which are x'' and p'', the displacement $o''y''$ of y may be found and in the same manner the displacements of all the remaining panel points of the truss. The complete displacement diagram for this truss is shown in (*b*) of Fig. 47.

PROBLEM

No. **46***a*. By means of a displacement diagram such as shown in (*b*) of Fig. 46, determine the horizontal and vertical motion of all the panel points of Fig. 44*c* of page 140 for the left end of the truss fixed in position and the right end on rollers. Make a drawing of the normal figure of the truss and show by coordinates the positions of the displaced panel points.

ART. 47. GRAPHICAL METHOD OF FINDING THE DISPLACEMENTS OF THE PANEL POINTS OF STRUCTURES WHICH ARE UNSYMMETRICALLY DEFORMED

Many structures are of such shape of figure, or varying cross-section of members, or the individual members may be so deformed, as to make it impossible to select a member which has no rotation in the deformed framework.

For such structures any member is first assumed to have no rotation, and a displacement diagram for the panel points of the frameworks made under this assumption, then the displacements first determined may be corrected to conform to the known conditions governing the motion of the structure as a whole.

Let the full line sketch of (*a*) of Fig. 47 be the truss of Fig. 44*c* of page 140. The displacement diagram (*b*) of this truss was made to illustrate the last article. Let it be assumed that the member sz undergoes no rotation and that the point s is fixed in position as it actually is. By the method of the preceding Article the displacement diagram is made and the displacement of the various panel points determined in accordance with the assumption.

The line $a''s''$ would be the displacement of the point a with reference to its former position, that is, the vertical motion is $a's''$ and the horizontal motion $a'a''$ if there is no rotation of the member sz. The vertical motion, $a's''=.53$ ft. here indicated, cannot take place, as it means that the point a is over 6″ above the support. The nature of the supports fixes the vertical deflection of a as 0. That is the displacement of the various panel points must be corrected for a rotation sufficient to bring this point down to the support.

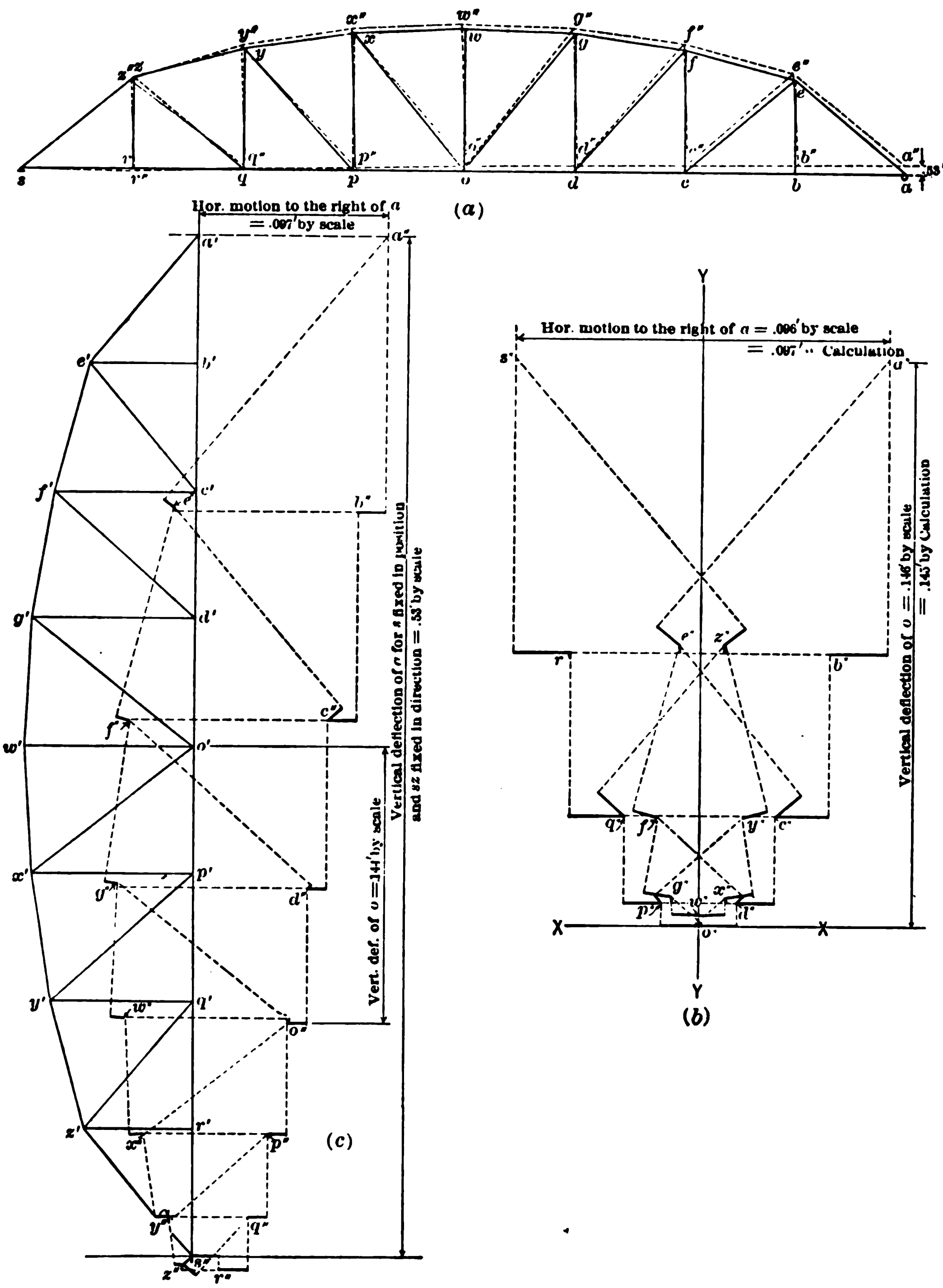

(a)
Hor. motion to the right of a
= .097' by scale
Vertical deflection of a for s fixed in position
and sz fixed in direction = .53' by scale
Vert. def. of o = .144' by scale
(c)
Hor. motion to the right of a = .096' by scale
= .097' .. Calculation
Vertical deflection of o = .146' by scale
= .143' by Calculation
(b)

FIG. 47.

The displacements of the various panel points for s fixed in position and sz fixed in direction, with reference to the normal positions, are shown by means of the dotted line sketch of (a) of Fig. 47. The panel points of the dotted sketch are located by observing that for the displacement diagram (c) the point r moves downward and to the right, and the remaining bottom chord points upward to the right. The point z moves downward and to the left, and all other upper chord points upward and to the left. These displaced positions are only approximately shown, as the actual displacements are so small that they cannot be accurately plotted to the scale of the truss drawing. It is seen that to get the true displacements of any panel point of the truss that its displacement shown in (c) must be corrected for the space passed through by this panel point of the dotted figure of (a) when the whole dotted line truss is rotated about s through the angle whose $\tan = \frac{.53}{152.097} = 0.0035$. As this angle is so very small, and as the rotation for any case which can occur in a practical problem is very small, the rotation may be taken, without appreciable error, as measured on a perpendicular at the panel point to the line joining the panel point and the center of rotation.

The corrections may be applied analytically or graphically, but a graphical correction is simplest.

The corrected displacement of a is given graphically by the line $a'a''$ of (c), for the vertical distance $s''a' = .53$ ft. has been taken from the displacement $s''a''$. The corrected displacement of o is the distance $o'o''$ for the displacement $s''o''$ has been corrected by taking the vertical distance so' $\left(=\frac{76}{152}\times .53 = .265 \text{ ft.}\right)$ from it, and so on for each of the bottom chord points.

The corrected displacement of w is the distance $w'w''$, as it is the distance $s''w''$ corrected by taking $s''w'$ $\left(=\frac{\sqrt{76^2+25^2}}{152}\times .53 \text{ ft.} = \frac{80}{152}\times .53 \text{ ft.} = .28 \text{ ft.}\right)$ from it. The remaining top chord displacements may be corrected in the same manner. It will be noted that the lines joining the points a', b', c', e', f', etc., from which the corrected displacements are to be measured, form a figure exactly similar to the normal truss, each member of which is perpendicular to the corresponding truss member. Hence to determine the points b', c', e', f', etc., after the point a' has been found, it is only necessary to make a figure having $s''a'$ as the length and similar to the original figure. Any other panel point and a member at that point could be taken instead of z and sz to find the final displacements, and any point other than a would lead to a smaller and simpler figure, as the rotations of the members at the other points are less than at a and s.

PROBLEM

No. **47***a*. Construct a displacement diagram for the truss of Fig. 44*c* of page 140 by assuming the point *a* fixed in position and the member *ar* fixed in direction, and correct the same for rotation, thus finding the true displacements.

ART. 48. COMPARISON OF THE ANALYTICAL AND GRAPHICAL METHODS OF DETERMINING THE DISPLACEMENTS OF THE PANEL POINTS OF A SMALL CANTILEVER

The analytical computation of the horizontal motion of *s* and the vertical motion of *k* has been made in table No. 48*a* for the truss of Fig. 48*a* by the method of work due to an auxiliary load of unity. The computation of the deflection for any other point can be quickly made as soon as the stresses due to a load of unity at the point are known. The condition of loading taken is that of live load over the cantilever arm and suspended span combined with full dead load, as this is the condition giving the greatest motions of the points selected.

It should be noted that each member is assumed to have a full pin bearing at each end, although this is not generally the case in practice. Where the deflections for several points of the same structure are desired, the change in length of each member should be determined for the given condition of loading and multiplied, with one setting of the slide rule, by the stresses due to each of the unit loads; thus giving the partial deflection at all the desired points, produced by the change in length of the given number. Where the points whose deflections are desired are in the cantilever arm and suspended span, it should be noted that the combined effect of the distortions of all the members of the anchor arm in producing deflection at the several points varies as the reactions at the anchor pier due to the unit loads at the points whose deflections are desired. For example the vertical deflection of *k* due to the members *ab* to *oe* inclusive $= .134375$ ft. and the effect of these same members produces a horizontal deflection at *s* of $.134375 \times \frac{.25}{.50} = .067188$ ft., and therefore it is not necessary to compute the individual quantities in the last column of the table for the members *ab* to *oe* inclusive, provided we have the values for these members in the ninth column of the table.

It may also be noted here that to correct the analytical computations for the deflection of a point due to a change in the distortion of one or more members of any truss, only the partial deflections for the members which have had their distortions changed need alteration. For deflections determined graphically any change in the distortion of one member requires the reconstruction of the entire

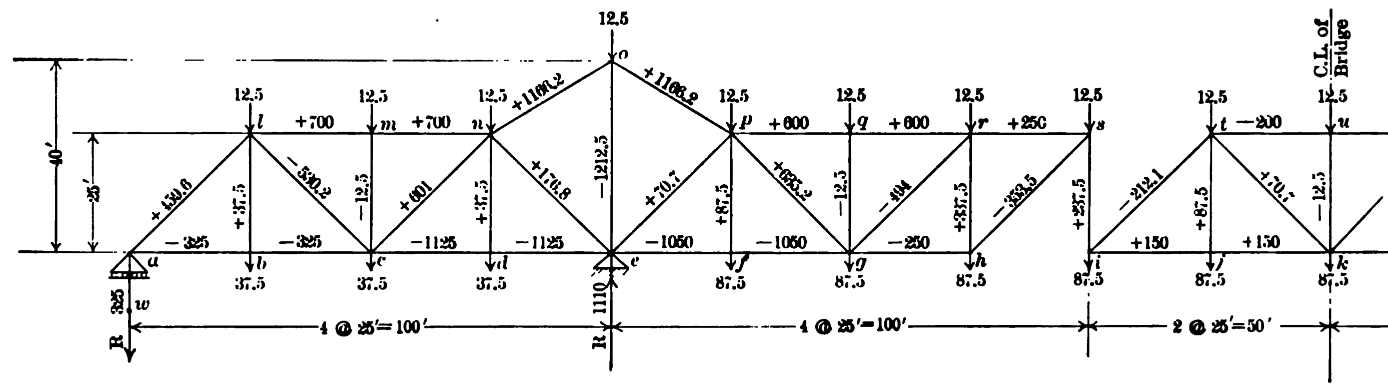


FIG. 48a.

TABLE No. 48a

Member.	Length in Feet.	Stresses in 1000 of Lbs.	Area Cross-Section in Sq. Ins.	Elastic Change in Length $SL/AE=\lambda_1$	Change Length due to Pin Holes $=\lambda_2$	$\lambda_1+\lambda_2=\lambda$	Stresses due Unit Load at $k(\downarrow)$ T_V	Partial Deflections λT_V. ft.	Stress due Unit Load at $s(\leftarrow)$ T_H	Partial Deflections λT_H. ft.
ab	25.000	− 325.0	40.63	−.006666	−.0026	−.009266	−0.5000	.004633	+0.2500	−.002316
al	35.354	+ 459.6	45.96	+.011785	+.0026	+.014385	+0.7071	.010172	−0.3535	−.005086
lb	25.000	+ 37.5	10.00	+.003125	+.0026	+.005725	0	0	0	0
bc	25.000	− 325.0	40.63	−.006666	−.0026	−.009266	−0.5000	.004633	+0.2500	−.002317
lc	35.354	− 530.2	66.27	−.009428	−.0026	−.012028	−0.7071	.008505	+0.3535	−.004252
lm	25.000	+ 700.0	70.00	+.008333	+.0026	+.010933	+1.0000	.010933	−0.5000	−.005467
mc	25.000	− 12.5	10.00	−.001042	−.0026	−.003642	0	0	0	0
mn	25.000	+ 700.0	70.00	+.008333	+.0026	+.010933	+1.0000	.010933	−0.5000	−.005467
cn	35.354	+ 601.0	60.10	+.011785	+.0026	+.014385	+0.7071	.010172	−0.3535	−.005086
de	25.000	−1125.0	140.63	−.006667	−.0026	−.009267	−1.5000	.013900	+0.7500	−.006950
cd	25.000	−1125.0	140.63	−.006667	−.0026	−.009267	−1.5000	.013900	+0.7500	−.006950
nd	25.000	+ 37.5	10.00	+.003125	+.0026	+.005725	0	0	0	0
ne	35.354	+ 176.8	17.68	+.011785	+.0026	+.014385	+0.3536	.005087	−0.1768	−.002543
no	29.155	+1166.2	116.62	+.009718	+.0026	+.012318	+1.4577	.017957	−0.7289	−.008979
aw	18.000	+ 325.0	32.50	+.006000	+.0013	+.007300	+0.5000	.003650	−0.2500	−.001825
oe	40.000	−1212.5	151.56	−.010667	−.0026	−.013267	−1.5000	.019900	+0.7500	−.009950

ep	35.354	+ 70.7	10.00	+.008333	+.0026	+.010933	+0.3536	.003866	−0.5303	−.005798
ef	25.000	−1050.0	131.25	−.006667	−.0026	−.009267	−1.5000	.013900	0	0
fg	25.000	−1050.0	131.25	−.006667	−.0026	−.009267	−1.5000	.013900	0	0
pf	25.000	+ 87.5	10.00	+.007293	+.0026	+.009893	0	0	0	0
pg	35.354	+ 635.2	63.52	+.011785	+.0026	+.014385	+0.7071	.010172	0	0
pq	25.000	+ 600.0	60.00	+.008333	+.0026	+.010933	+1.0000	.010933	−1.0000	−.010933
qr	25.000	+ 600.0	60.00	+.008333	+.0026	+.010933	+1.0000	.010933	−1.0000	−.010933
qg	25.000	− 12.5	10.00	−.001042	−.0026	−.003642	0	0	0	0
gr	35.354	− 494.0	61.75	−.009428	−.0026	−.012028	−0.7071	.008505	0	0
rh	25.000	+ 337.5	33.75	+.008333	+.0026	+.010933	+0.5000	.005467	0	0
gh	25.000	− 250.0	31.25	−.006667	−.0026	−.009267	−0.5000	.004633	0	0
rs	25.000	+ 250.0	25.00	+.008333	+.0026	+.010933	+0.5000	.005467	−1.0000	−.010933
hs	35.354	− 353.5	44.19	−.009428	−.0026	−.012028	−0.7071	.008505	0	0
hi	25.000									
op	29.155	+1166.2	116.62	+.009718	+.0026	+.012318	+1.4577	.017957	−0.7289	−.008979
si	25.000	+ 237.5	23.75	+.008333	+.0026	+.010933	+0.5000	.005466	0	0
it	35.354	− 212.1	26.51	−.009428	−.0026	−.012028	−0.7071	.008505	0	0
ij	25.000	+ 150.0	15.00	+.008333	+.0026	+.010933	+0.5000	.005466	0	0
jk	25.000	+ 150.0	15.00	+.008333	+.0026	+.010933	+0.5000	.005466	0	0
tj	25.000	+ 87.5	10.00	+.007293	+.0026	+.009893	0	0	0	0
tu	25.000	− 200.0	25.00	−.006666	−.0026	−.009266	−1.0000	.009266	0	0
tk	35.354	+ 70.7	10.00	+.008333	+.0026	+.010933	+0.7071	.007731	0	0
uk	25.000	− 12.5	10.00	−.001042	−.0026	−.003642	0	0	0	0
st	25.000									
								.289513		−.114764

Total vertical deflection at $k=2\times0.289513=0.579026$ ft. $=6.949$ ins.

Total horizontal deflection at $s=1\times-0.114764=-.114764$ ft. $=-1.377$ ins.

portion of the diagram, which depends on the member with the changed distortion.

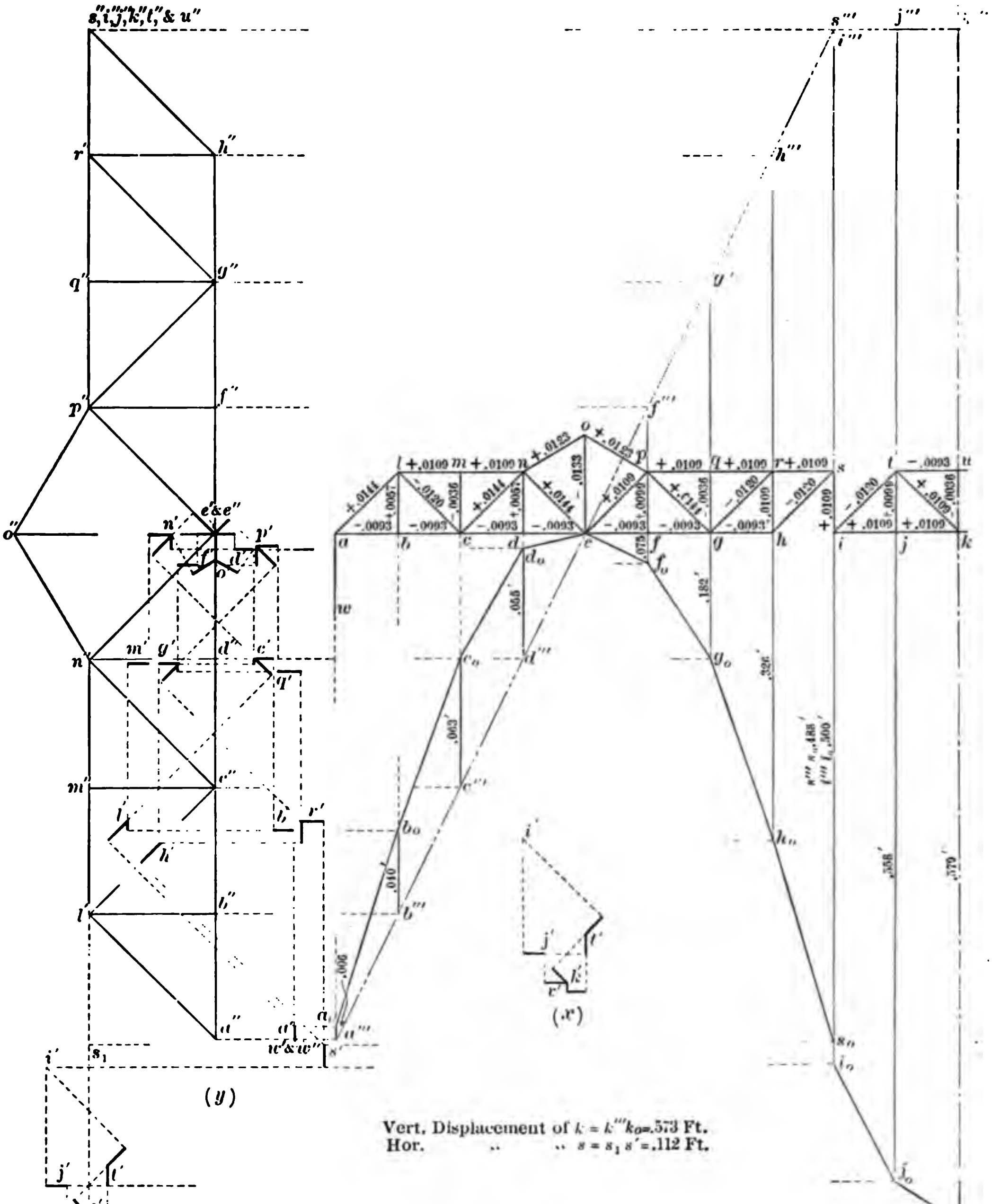


FIG. 48b.

The graphical determination of the displacements of the various panel points of the truss of Fig. 48*a* due to the changes in length λ given in column No. 7 of Table No. 48*a* is shown in Fig. 48*b*.

The displacement diagram y is made by taking e as the fixed point, as it is, and assuming the direction of oe fixed.

This gives the line joining e' and a' as the displacement of a.

It is known from the nature of the support at the anchor pier that the vertical motion of a is only that of the stretch in the anchor bars, and the horizontal motion is, for such small rotation as takes place in oe, the horizontal projection of $e'a'$.

Therefore the point a'' of the rotation diagram is determined, as it is the only point which gives the known displacement $a''a'$ of the point a. Knowing that the actual rotation of any other point of the truss is to the rotation of a as its distance to the center of rotation of the entire truss is to a's distance from the same center, the remainder of the rotation diagram is made.

The total displacements of the panel points of the suspended span are found by taking the point k as fixed in position and the member ku fixed in direction and considering the suspended span as an independent structure as is shown in x of Fig. 48*b*. As ku is vertical in the distorted figure the rotation diagram is a point, and to show the full motion of all its points the figure x is redrawn in connection with y.

The final motion of any panel point is given in y, the points marked with the primes showing this motion with reference to the corresponding point marked with the seconds. The graphical method of finding deflections is particularly useful in connection with Maxwell's theorem in determining statically indeterminate stresses in certain structures. This special use is well illustrated in the case of the two-hinged arch with open framework.

For a full development of the graphical method of finding the displacements of open frameworks, see a paper in the Journal of the Association of Engineering Societies by David Molitor, read May 16, 1894, before the Engineers' Club of St. Louis. This paper is the first comprehensive treatment of the subject published in America, and in it credit is given to those who are principally responsible for the development of the method.

PROBLEMS

No. **48*a***. Find the horizontal motion of o, the top of the post over the main pier, analytically from the data of the table.

No. **48*b***. Find the horizontal motion of h, the outer end of the bottom chord of the cantilever arm, analytically from the data of the table.

No. **48*c***. Make a graphical determination of the displacements of the panel points of the truss of Fig. 48*a* for the distortion λ of each member, by taking e as the fixed point and assuming ed fixed in direction and then correcting for rotation. Make a table showing the horizontal and vertical motion of each panel point of the bridge.

ART. 49. THE AMOUNT OF THE DEFORMATIONS OF THE MEMBERS OF A TRUSS AS AFFECTING THE RELIABILITY OF THE DIFFERENT METHODS FOR FINDING ITS DEFLECTIONS

The distortions which a truss undergoes may be determined with reliability, for large distortions of its separate members, by only one method, that of geometry. All methods of determining deflections, other than that of geometry, are based on the assumption of close similarity between the normal and the deformed truss or, in other words, on the assumption that only infinitesimal deformations occur. This assumption is only approximately true, but fortunately, for the problems of practice, actual deformations for structures which may be used in engineering works are always very small.

That the student may see for himself the limitation of the methods and know also that proportionality of strain to stress, for the individual members of a truss,

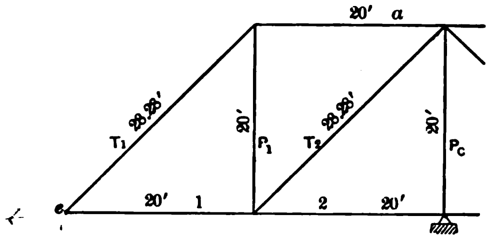


FIG. 49*a*.

does not necessarily imply the same relation between deformation and load for the structure as a whole, the deflection of point *e* of one arm of the cantilever truss of Fig. 49*a* will be determined for two cases: Case I, where the stresses and sections of the members are such as to produce changes in length of the members of $\frac{1}{20}$ the normal; Case II, where the changes in length are twice the amount for Case I.

Fig. 49*b* is a redrawing of the truss of Fig. 49*a*, using the new lengths of the members as they are stated on the figure, and Table No. 49*a* is an analytical computation of the horizontal and vertical motions of *e* by the method of work due to an auxiliary load of unity for the changes in length of Case I.

Fig. 49*c* and Table No. 49*b* are the same graphical construction and analytical computations for the changes in length of Case II.

Case I. All members change $\frac{1}{20}$ of their length, tension members being lengthened and compression members being shortened by this amount.

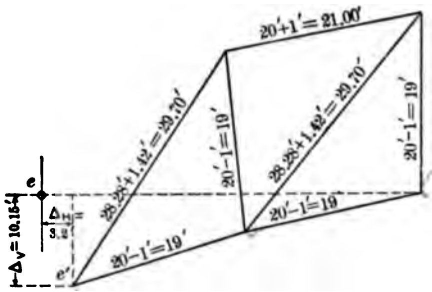


Fig. 49b

TABLE No. 49a

Member.	λ	Stresses Due to Unity at e		Computed partial Vert. Def.	Computed partial Horiz. Def.
		Vert.	Horiz.		
T_1	+1.42	+1.41	0	+2.00	0
P_1	−1.00	−1.00	0	+1.00	0
T_2	+1.42	+1.41	0	+2.00	0
Pc	−1.00	−1.00	0	+1.00	0
a	+1.00	+1.00	0	+1.00	0
1	−1.00	−1.00	−1.00	+1.00	1.00
2	−1.00	−2.00	−1.00	+2.00	1.00
				$\Sigma\Delta_V = 10.00'$, $\Delta_H = 2.00'$.	

Case II. All members change $\frac{1}{10}$ of their length.

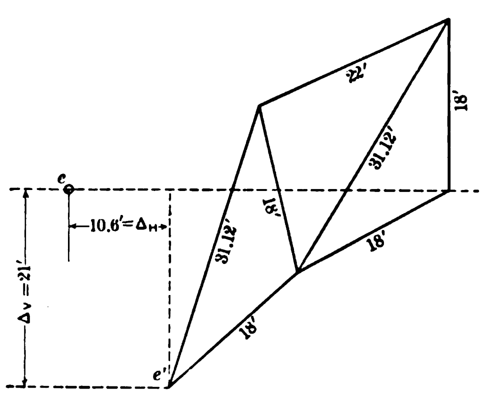


Fig. 49c

TABLE No. 49b

Member.	λ	Stresses Due to Unity at e		Computed partial Vert. Def.	Computed partial Horiz. Def.
		Vert.	Horiz.		
T_1	+2.84	+1.41	0	4.00	0
P_1	−2.00	−1.00	0	2.00	0
T_2	+2.84	+1.41	0	4.00	0
Pc	−2.00	−1.00	0	2.00	0
a	+2.00	+1.00	0	2.00	0
1	−2.00	−1.00	−1.00	2.00	2.00
2	−2.00	−2.00	−1.00	4.00	2.00
				$\Sigma\Delta_V = 20.00$, $\Delta_H = 4.00$.	

Table 49*b* shows deflections twice as great as for Table 49*a*, as it must. Fig. 49*c*, however, does not show deflections bearing any definite relation to those of Fig. 49*b*. While these graphical constructions have not been made with special care, they are more accurate for such changes as have been assumed than the analytical computations.

Elastic changes in length of the members of a truss of as much as $\frac{1}{10}$ are impossible for a safe structure of any of the common engineering materials, and for distortions of the usual amount the methods previously outlined for finding deflections are entirely reliable.

In the effort to build very long span bridges with a material having a modulus of elasticity of not over 30,000,000, it should not be forgotten that a unit stress as high as 40,000 lbs. per square inch would give excessive distortion to the truss.

ART. 50. STRESSES IN REDUNDANT MEMBERS BY MEANS OF CASTIGLIANO'S SECOND THEOREM

The Second Theorem of Castigliano gives a method for finding the stresses in the one or more redundant members which a truss or structure may contain. As a first example of the application of the method to finding the stress in such a member, take the truss of Fig. 50*a* having two diagonal members P_L and P_R in the center panel. Let it be required to find the stresses in P_L and P_R, which stresses will be designated by the mark of the member in which they occur, due to two loads of 150,000 lbs. placed as shown, this being approximately the loading producing

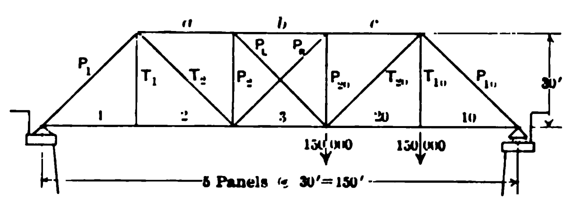


FIG. 50*a*.

maximum shear in the center panel. The member P_R will be considered cut at its center and the remaining truss designated as the static figure. With the member P_R cut at its center the cut ends would, in general, have a motion in the direction of the length of the member; and, it may be stated, from the Second Theorem of Castigliano, that this motion, $\Delta = \frac{dK}{dP_R}$. To find the stress in P_R when it is not cut it is only necessary to form $\frac{dK}{dP_R}$ and place it equal to zero, that is, make the motion of the cut ends of P_R equal to zero when cut, and solve for P_R. It is readily seen that the true stresses, those existing when P_R is acting, are those for the truss with P_R considered cut, modified by the stress in P_R necessary to bring the cut ends together. The following table shows how the computation may be conveniently arranged.

The value of $\frac{dK}{dP_R}$ following the table is obtained by adding the parts of column No. 9 of the table. Placing $\frac{dK}{dP_R} = 0$ and solving, P_R is found to be 61,300 pounds.

Substituting this value of P_R in the seventh column of the table the last column is prepared.

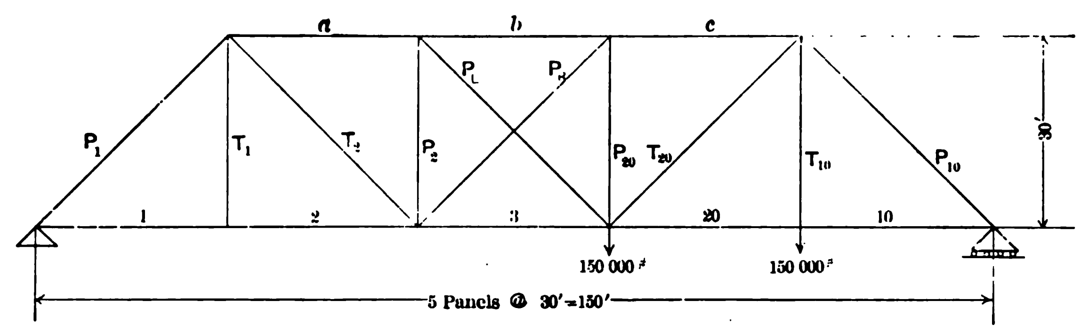


FIG. 50a.

TABLE No. 50a

Members	Length in Feet. L	Area in Sq. ins. A	$\frac{L}{A}$	Stresses in the Static Figure for the Given Loads.	Stresses due to P_R	Total Stresses.	Partial Values of K = Work in Each Member.	$\frac{dK}{dP_R}$	True Stresses for given Loading.
P_1	42.426	42.0	1.010	-127300	0	-127300		0	-127300
T_1	30.000	14.0	2.143	0	0	0		0	0
T_2	42.426	14.0	3.030	$+127300$	0	$+127300$		0	$+127300$
P_2	30.000	9.0	3.333	-90000	$+.707P_R$	$-90000+.707P_R$	$\frac{L}{2EA}[-90000+.707P_R]^2$	$\frac{1}{E}[-212100+1.667P_R]$	-46700
P_L	42.426	12.0	3.535	$+127300$	$-P_R$	$+127300-P_R$	$\frac{L}{2EA}[+127300-P_R]^2$	$\frac{1}{E}[-450000+3.535P_R]$	$+66000$
P_R	42.426	12.0	3.535	0	$-P_R$	$+0-P_R$	$\frac{L}{2EA}[-P_R]^2$	$\frac{1}{E}[0+3.535P_R]$	-61300
P_{20}	30.000	9.0	3.333	0	$+.707P_R$	$+0+.707P_R$	$\frac{L}{2EA}[+.707P_R]^2$	$\frac{1}{E}[0+1.666P_R]$	$+43300$
T_{20}	42.426	14.0	3.030	$+84800$	0	$+84800$		0	$+84800$
T_{10}	30.000	14.0	2.143	$+150000$	0	$+150000$		0	$+150000$
P_{10}	42.426	42.0	1.010	-297000	0	-297000		0	-297000
a	30.000	39.0	0.769	-180000	0	-180000		0	-180000
b	30.000	39.0	0.769	-270000	$+.707P_R$	$-270000+.707P_R$	$\frac{L}{2EA}[-270000+.707P_R]^2$	$\frac{1}{E}[-146800+0.385P_R]$	-226700
c	30.000	39.0	0.769	-270000	0	-270000		0	-270000
1	30.000	22.0	1.364	$+90000$	0	$+90000$		0	$+90000$
2	30.000	22.0	1.364	$+90000$	0	$+90000$		0	$+90000$
3	30.000	32.0	0.937	$+180000$	$+.707P_R$	$+180000+.707P_R$	$\frac{L}{2EA}[+180000+.707P_R]^2$	$\frac{1}{E}[+119300+0.468P_R]$	$+223300$
20	30.000	22.0	1.364	$+210000$	0	$+210000$		0	$+210000$
10	30.000	22.0	1.364	$+210000$	0	$+210000$		0	$+210000$
							$\Sigma =$	$\frac{1}{E}[-689600+11.256P_R]$	

$\frac{1}{E}[-68960+11.256P_R]=0;\qquad 11.256P_R=68960;\qquad P_R=\frac{68960}{11.256}=61300.$

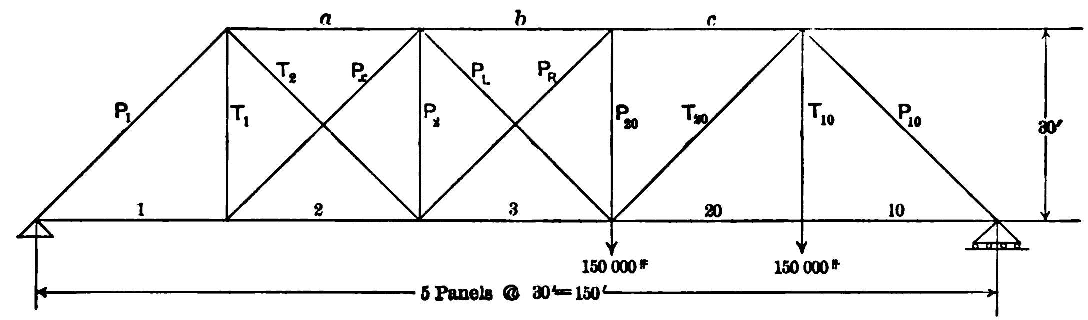


FIG. 50b.

TABLE No. 50b

Members.	$\frac{L}{A}$	Stresses in the Static Figures for the Given Loads.	Stresses due to P_R.	Stresses due to P_x.	Total Stresses.	Partial Values of K = Work in Each Member.	$\frac{dK}{dP_R}$	$\frac{dK}{dP_x}$	True Stresses for given Loading.
P_1	1.010	-127300	0	0	-127300		0	0	-127300
T_1	2.143	0	0	$+.707P_x$	$+.707P_x$	$\frac{L}{2EA}[.707P_x]^2$	0	$\frac{1}{E}[\quad +1.072P_x]$	$+\ 37300$
T_2	3.030	$+127300$	0	$-P_x$	$+127300-P_x$	$\frac{L}{2EA}[+127300-P_x]^2$	0	$\frac{1}{E}[385700 \quad +3.030P_x]$	$+\ 74600$
P_x	3.030	0	0	$-P_x$	$-P_x$	$\frac{L}{2EA}[-P_x]^2$	0	$\frac{1}{E}[\quad +3.030P_x]$	$-\ 52700$
P_2	3.333	$-\ 90000$	$+.707P_R$	$+.707P_x$	$-\ 90000$ $+.707[P_R+P_x]$	$\frac{L}{2EA}[-\ 90000$ $+.707(P_R+P_x)]^2$	$\frac{1}{E}[-212100+1.667P_R$ $+1.667P_x]$	$\frac{1}{E}[-212100+1.667P_R$ $+1.667P_x]$	$-\ 14900$
P_L	3.535	$+127300$	$-P_R$	0	$+127300-P_R$	$\frac{L}{2EA}[+127300-P_R]^2$	$\frac{1}{E}[-450000+3.535P_R]$	0	$+\ 73800$
P_R	3.535	0	$-P_R$	0	$-P_R$	$\frac{L}{2EA}[-P_R]^2$	$\frac{1}{E}[\quad +3.535P_R]$	0	$-\ 53500$

P_{20}	3.333	0	$+.707P_R$	0	$+.707P_R$	$\frac{L}{2EA}[+.707P_R]^2$	$\frac{1}{E}[\quad +1.666P_R]$	0	+ 37800
T_{20}	3.030	+ 84800	0	0	+ 84800		0	0	+ 84800
T_{10}	2.143	+150000	0	0	+150000		0	0	+150000
P_{10}	1.010	−297000	0	0	−297000		0	0	−297000
a	0.769	−180000	0	$+.707P_x$	$-180000 +.707P_x$	$\frac{L}{2EA}[-180000 +.707P_x]^2$	0	$\frac{1}{E}[-\ 97900 \ +0.384P_x]$	−142700
b	0.769	−270000	$+.707P_R$	0	$-270000 +.707P_R$	$\frac{L}{2EA}[-270000 +.707P_R]^2$	$\frac{1}{E}[-146800+0.385P_R]$	0	−232200
c	0.769	−270000	0	0	−270000		0	0	−270000
1	1.364	+ 90000	0	0	+ 90000		0	0	+ 90000
2	1.364	+ 90000	0	$+.707P_x$	$+\ 90000 +.707P_x$	$\frac{L}{2EA}[+\ 90000 +.707P_x]^2$	0	$\frac{1}{E}[+\ 86800 \ +0.682P_x]$	+127300
3	0.937	+180000	$+.707P_R$	0	$+180000 +.707P_R$	$\frac{L}{2EA}[+180000 +.707P_R]^2$	$\frac{1}{E}[+119300+0.468P_R]$	0	+217800
20	1.364	+210000	0	0	+210000		0	0	+210000
10	1.364	+210000	0	0	+210000		0	0	+210000
						$\Sigma =$	$\frac{1}{E}[-689600+11.256P_R +1.667P_x]$	$\frac{1}{E}[-608900 \ +1.667P_R +9.865P_x]$	

$$11.256P_R+1.667P_x= \ 689600$$
$$1.667P_R+9.865P_x= \ 608900$$

$$-P_R-0.148P_x=- \ 61260$$
$$+P_R+5.918P_x= +365260$$

$$+5.770P_x= +304000$$
$$P_x= \ 52700$$
$$P_R= \ 53500$$

In preparing the above table as far as the stress in P_R is concerned, it would have been enough to include only those members in which stress is produced by the variable stress, P_R, and therefore for many of the columns of the table only those members involving P_R, are tabulated.

The previous method is readily applicable to determining the stresses in any number of redundant members by simply considering the stresses in each redundant member as varying and forming a derivative of the work with respect to each varying stress and placing it equal to zero, thus writing as many equations as there are varying or statically indeterminate stresses, from which the unknown stresses may be found.

To illustrate, let another diagonal member P_x be introduced in the panel containing T_2 of the truss of Fig. 50*a*, of the same area as T_2 and crossing it at the center, thus forming the truss shown in Fig. 50*b* which heads Table No. 50*b*.

Computation for the stresses in the redundant members Px and P_R and also for the remaining truss members is shown in Table No. 50*b*.

PROBLEM

No. **50*a*.** Find the stresses in the members T_2, P_x, P_L and P_R for the truss just considered when loaded with 150,000 lbs. at the bottom of T_{10}, P_{20}, and P_2 by means of Castigliano's Second Theorem.

ART. 51. STRESSES IN REDUNDANT MEMBERS BY MEANS OF THE DISTORTION DUE TO UNIT LOADS ACTING IN THE REDUNDANT MEMBERS

While the previous article gives a very elegant method, from a theoretical standpoint, for determining the stresses in redundant members, it is somewhat simpler in the problems of practice to determine the stresses in redundant members in

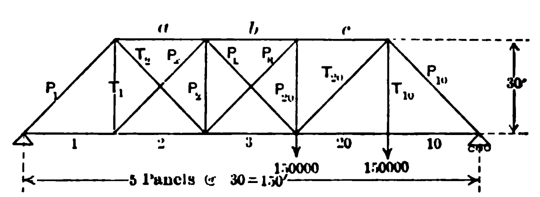


FIG. 51*a*.

terms of certain deflections of the static figure, the static figure being the original figure with enough bars considered cut to render it statically determinate. For the purpose of illustration the stresses in the members P_x and P_R of Fig. 51*a*, which is the truss of the previous article, will be determined, and from them the stresses in the entire truss. If P_x and P_R be considered cut at their centers, the resulting truss is statically determinate for the remaining members.

It is clear that the stresses in the members of the truss when the truss acts as a whole will be those of the static figure modified by the stresses in P_x and P_R, produced by bringing the cut ends together. In order to determine the stresses in P_x and P_R the following notation will be used:

Let S_{px} be the stress in P_x when the truss is acting as a whole;

S_{pr} be the stress in P_R when the truss is acting as a whole;

Δ_1 be the overlapping of the cut ends of P_x in the static figure;

Δ_2 be the overlapping of the cut ends of P_R in the static figure;

d_1 be the separation of the cut ends of P_x due to a force of unity acting toward the cut ends of P_x;

d_2 be the separation of the cut ends of P_R due to a force of unity acting towards the cut ends of P_R;

d_{10} be the separation of the cut ends of P_x due to a force of unity acting towards the cut ends of P_R;

d_{20} be the separation of the cut ends of P_R due to a force of unity acting towards the cut ends of P_x;

$d_{10} = d_{20}$ from Maxwell's Theorem.

Assuming that an analytical determination of the above defined deflections has been made, the following equations may be written:

For the member P_x,

$$\Delta_1 - S_{px}d_1 - S_{pr}d_{10} = 0. \quad \ldots \ldots \ldots \quad (1)$$

and for P_R

$$\Delta_2 - S_{pr}d_2 - S_{px}d_{10} = 0. \quad \ldots \ldots \ldots \quad (2)$$

Solving these equations it is found that

$$S_{px} = \frac{\Delta_2 . d_{10} - \Delta_1 . d_2}{d_{10}^2 - d_1 . d_2}, \quad \ldots \ldots \ldots \quad (3)$$

$$S_{pr} = \frac{\Delta_2 . d_1 - \Delta_1 . d_{10}}{d_1 . d_2 - d_{10}^2}, \quad \ldots \ldots \ldots \quad (4)$$

The computations of Table No. 51a show that the direction assumed for the various deflections in writing Eqs. (1) and (2) are correct, therefore: Substituting the values, without regard to signs, for Δ_1, Δ_2, d_1, d_2, d_{10}, and d_{20} in Eqs. (3) and (4), it is seen that:

$$S_{px} = \frac{689{,}600 \times 1.666 - 608{,}900 \times 11.256}{1.666 \times 1.666 - 9.864 \times 11.256} = \frac{-570{,}500}{-108.242} = 52{,}700.$$

$$S_{pr} = \frac{689{,}600 \times 9.864 - 608{,}900 \times 1.666}{9.864 \times 11.256 - 1.666 \times 1.666} = \frac{5{,}788{,}200}{108.242} = 53{,}500.$$

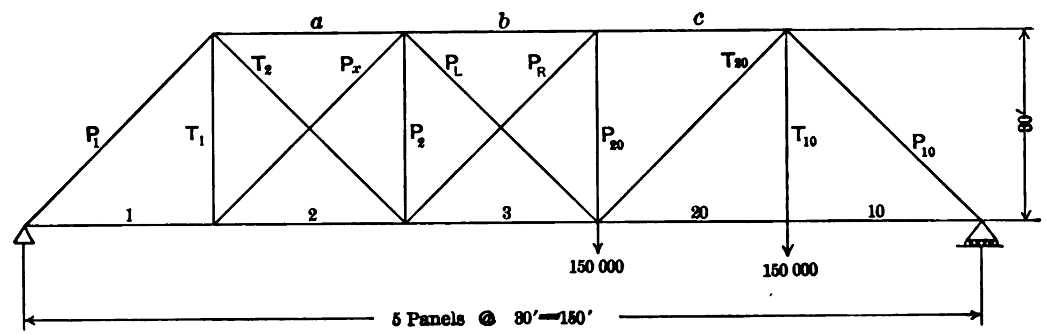


FIG. 51a

TABLE No. 51a

THE FOLLOWING IS A DETERMINATION OF THE PREVIOUSLY DEFINED DEFLECTIONS ASSUMING THAT $E=1$.

Member.	L Length. Ft.	A Area Sq. in.	$\frac{L}{A}$	S Stress in Static Fig.	T_1 Due unit Com. in P_x.	T_2 Due unit Com. in P_R.	$\Delta_1=\frac{ST_1L}{A}$	$\Delta_2=\frac{ST_2L}{A}$	$d_1=\frac{T_1^2L}{A}$	$d_2=\frac{T_2^2L}{A}$	$d_{10}=d_{20}=\frac{T_1T_2L}{A}$	Actual Stress.
P_1	42.426	42.0	1.010	−127300								−127300
T_1	30.000	14.0	2.143		+0.707				+1.071			+ 37300
T_2	42.426	14.0	3.030	+127300	−1.000		−385700		+3.030			+ 74600
P_x	42.426	14.0	3.030		−1.000				+3.030			− 52700
P_2	30.000	9.0	3.333	− 90000	+0.707	+0.707	−212100	−212100	+1.666	+1.666	+1.666	− 15000
P_L	42.426	12.0	3.535	+127300		−1.000		−450000		+3.535		+ 73800
P_R	42.426	12.0	3.535			−1.000				+3.535		− 53500
P_{20}	30.000	9.0	3.333			+0.707				+1.666		+ 37800
T_{20}	42.426	14.0	3.030	+ 84800								+ 84800
T_{10}	30.000	14.0	2.143	−150000								−150000
P_{10}	42.426	42.0	1.010	−296900								−296900
a	30.000	39.0	.769	−180000	+0.707		− 97900		+0.385			−142700
b	30.000	39.0	.769	−270000		+0.707		−1 6800		+0.385		−232200
c	30.000	39.0	.769	−270000								−270000
1	30.000	22.0	1.364	+ 90000								+ 90000
2	30.000	22.0	1.364	+ 90000	+0.707		+ 86800		+0.682			+127300
3	30.000	32.0	.937	+180000		+0.707		+119300		+0.469		+217800
20	30.000	22.0	1.364	+210000								+210000
10	30.000	22.0	1.364	+210000								+210000
						Sum = Σ =	−608900	−689600	+9.864	+11.256	+1.666	

With these values of S_{px} and S_{pr} the last column of the table is prepared. If the preceding truss contained only one superfluous bar, P_R, then

$$\Delta_2 - S_{pr}.d_2 = 0 \qquad \text{and} \qquad S_{rr} = \frac{689{,}600}{11{,}256} = 61{,}300,$$

as was found in the first part of the previous article.

PROBLEM

No. 51a. Find the stresses in the members T_7, P_x, P_L, and P_R for the truss just considered when loaded with 150,000 lbs. at the bottom of T_{10}, P_{30}, and P_3 by means of the method of this article.

ART. 52. FORMULÆ FOR DETERMINING THE STRESSES IN PARTIALLY CONTINUOUS TRUSSES

The ordinary cantilever bridge with a suspended span has certain advantages over other types which are too well known to be enumerated here. It also has certain disadvantages—one of which is the longitudinal swinging of the suspended span under the passage of live load. This swinging is produced by the alternate lowering of one support of the suspended span with reference to the other as a load travels over the bridge. Another of the disadvantages of this type is the expense of the large and powerful adjusting devices required to make the final connection at the center when the bridge is erected.

These disadvantages of the cantilever bridge may be overcome by omitting the suspended span entirely, as was done in the well known Queensborough Bridge of New York City. Fig. 52a shows the general arrangement of the spans and the nature of the supports of this bridge.

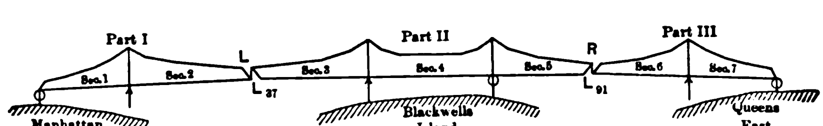


FIG. 52a.

The members L and R, of Fig. 52a, which connect the two cantilever arms over the left and right openings respectively, will be called rocker arms. It can readily be seen that if the stresses in these rocker arms are known for any given load, the computation of the stresses in all the truss members can be made as for a statically determined figure.

Let P_L = stress in the left rocker arm due to any loading.

Let P_R = stress in the right rocker arm due to any loading.

The deflections defined in the following are for both the rocker arms disconnected at their bottom ends.

Let d_1 = deflection of point L_{37} of the Manhattan lever arm, due to a load of unity at L_{37};

d_2 = deflection of point L_{37} (the bottom of the left rocker arm) of the Blackwell's Island west lever arm, due to a load of unity at L_{37};

d_3 = deflection of point L_{91} (the bottom of the right rocker arm) of the Blackwell's Island east lever arm, due to a load of unity at L_{91};

d_4 = deflection of point L_{91} of the Queens lever arm, due to a load of unity at L_{91};

d_{20} = deflection of point L_{37} of the Blackwell's Island west lever arm, due to a load of unity at L_{91};

d_{30} = deflection of point L_{91} of the Blackwell's Island east lever arm, due to a load of unity at L_{37};

D_1 = deflection of point L_{37} of the Manhattan lever arm, due to the given loading;

D_2 = deflection of point L_{37} of the Blackwell's Island west lever arm, due to the given-loading;

D_3 = deflection of point L_{91} of the Blackwell's Island east lever arm, due to the given loading.

D_4 = deflection of point L_{91} of the Queens lever arm, due to the given loading.

The above deflections it will be noted are all for bottom chord points.

The deflections for the Manhattan and Queens anchor and lever arms should always include that due to the anchor bars.

The deflections for the Blackwell's Island lever arms should include that due to the rocker arms.

The rocker arms L and R if disconnected at their bottom ends divide the structure into three portions, which are each statically determinate. These three portions will be designated as Part I, Part II and Part III.

Part I will be divided into Section No. 1, the Manhattan Anchor Arm, and Section No. 2, the Manhattan Lever Arm.

Part II will be divided into Section No. 3, the Blackwell's Island West Lever Arm, Section No. 4, the Blackwell's Island Span, and Section No. 5, the Blackwell's Island East Lever Arm.

Part III will be divided into Section No. 6, the Queens Lever Arm, and Section No. 7, the Queens Anchor Arm.

The adjacent lever arms of Parts I and II when the rocker arm L is connected move up and down together and the bottom end of L, which is a common point for both parts, has the same motion.

The adjacent lever arms of Parts II and III when the rocker arm R is connected move up and down together, and the bottom end of R which is a common point for both parts has the same motion.

With a live load on Part I and L and R connected, the members of Part I can only receive stress from the load and from the rocker arm stress P_L produced by this load, the members of Part II can only receive stress from the stresses P_L and P_R and the members of Part III can only receive stress from the stress P_R.

For a live load on Part II:

The stresses in Part I are due to P_L.

The stresses in Part II are due to the load and P_L and P_R.

The stresses in Part III are due to P_R.

For a live load on Part III:

The stresses in Part I are due to P_L.

The stresses in Part II are due to P_L and P_R.

The stresses in Part III are due to P_R and the load.

A careful analytical determination of the several previously defined deflections should now be made. For the present case such computation shows that:

A vertical downward force of unity will make d_1, d_2, d_3, d_4, d_{20} and d_{30}, all downward.

A vertical downward live load in Section No. 1 will make D_1, upward.

A vertical downward live load in Section No. 2 will make D_1, downward.

A vertical downward live load in Section No. 3 will make D_2, downward, and D_3, downward.

A vertical downward live load in Section No. 4 will make D_2, upward, and D_3, upward.

A vertical downward live load in Section No. 5 will make D_2, downward, and D_3, downward.

A vertical downward live load in Section No. 6 will make D_4, downward.

A vertical downward live load in Section No. 7 will make D_4, upward.

It is now possible to write equations for the stresses P_L and P_R in terms of known quantities (P_L and P_R will be taken as in tension or plus until their sign is known).

Take first a load on Section No. 1, for which it is now known that the motion of the bottom ends of both L and R is upward when the structure is acting as a whole.

The upward motion of the bottom end of L for section No. 2 is $D_1+P_L.d_1$.

The upward motion of the bottom end of L for section No. 3 is $-P_L.d_2-P_R.d_{20}$.

These must be equal.

$$\therefore D_1+P_L.d_1=-P_L.d_2-P_R.d_{20} \text{ or } P_L(d_1+d_2)=-D_1-P_R.d_{20} \quad . \quad . \quad (1a)$$

The upward motion of the bottom end of R for section No. 5 is $-P_R.d_3-P_L.d_{30}$.

The upward motion of the bottom end of R for section No. 6 is $P_R.d_4$.

These must be equal.

$$\therefore -P_R.d_3-P_L.d_{30}=P_R.d_4 \text{ or } P_R(d_3+d_4)=-P_L.d_{30} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

Take now a load on section No. 2. The motion of the bottom ends of both L and R is downward.

The downward motion of the bottom end of L for section No. 2 is $D_1-P_L.d_1$.

The downward motion of the bottom end of L for section No. 3 is $P_L.d_2+P_R.d_{20}$.

These must be equal.

$$\therefore D_1-P_L.d_1=P_L.d_2+P_R.d_{20} \text{ or } P_L(d_1+d_2)=D_1-P_R.d_{20} \quad . \quad . \quad . \quad (2a)$$

The downward motion of the bottom end of R for section No. 5 is $P_R.d_3+P_L.d_{30}$.

The downward motion of the bottom end of R for section No. 6 is $-P_R.d_4$.

These must be equal.

$$\therefore P_R.d_3+P_L.d_{30}=-P_R.d_4 \text{ or } P_R(d_3+d_4)=-P_L.d_{30} \quad . \quad . \quad . \quad (2b)$$

Take now a load on section No. 3. The motion of the bottom ends of L and R is downward.

The downward motion of the bottom end of L for section No. 2 is $-P_L.d_1$.

The downward motion of the bottom end of L for section No. 3 is $D_2+P_L.d_2+P_R.d_{20}$.

These must be equal.

$$\therefore -P_Ld_1=D_2+P_L.d_2+P_R.d_{20} \text{ or } P_L(d_1+d_2)=-D_2-P_R.d_{20}. \quad . \quad (3a)$$

The downward motion of the bottom end of R for section No. 5 $=D_3+P_R.d_3+P_L.d_{30}$.

The downward motion of the bottom end of R for section No. 6 is $-P_R.d_4$.

These must be equal.

$$\therefore D_3+P_R.d_3+P_L.d_{30}=-P_R.d_4 \text{ or } P_R(d_3+d_4)=-D_3-P_L.d_{30} \quad . \quad (3b)$$

Take now a load on section No. 4. The motion of the bottom ends of L and R is upward.

The upward motion of the bottom end of L for section No. 2 is $P_L.d_1$.

The upward motion of the bottom end of L for section No. 3 is $D_2-P_L.d_2-P_R.d_{20}$.

These must be equal.

$$\therefore P_L.d_1=D_2-P_L.d_2-P_R.d_{20} \text{ or } P_L(d_1+d_2)=D_2-P_R.d_{20}. \quad . \quad . \quad (4a)$$

The upward motion of the bottom end of R for section No. 5 is $D_3-P_R.d_3-P_L.d_{30}$.

The upward motion of the bottom end of R for section No. 6 is $P_R.d_4$.

These must be equal.

$$\therefore D_3-P_R.d_3-P_L.d_{30}=P_R.d_4 \text{ or } P_R(d_3+d_4)=D_3-P_L.d_{30}. \quad . \quad . \quad (4b)$$

For a load on each of the three remaining sections two simultaneous equations for finding P_L and P_R may be written.

For a load on section No. 5, they are:

$$P_L(d_1+d_2)=-D_2-P_R.d_{20}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

$$P_R(d_3+d_4)=-D_3-P_L.d_{30}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

For a load on section No. 6, they are:

$$P_L(d_1+d_2)=-P_R.d_{20}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

$$P_R(d_3+d_4)=D_4-P_L.d_{30}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

For a load on section No. 7, they are:

$$P_L(d_1+d_2)=-P_R.d_{20}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

$$P_R(d_3+d_4)=-D_4-P_L.d_{30}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

The above seven pairs of equations may be very much simplified by solving them for P_L and P_R.

The solved equations are:

For a load on section No. 1, $P_L=-M.D_1$.

$$P_R=+N.D_1.$$

For a load on section No. 2, $P_L=+M.D_1$.

$$P_R=-N.D_1.$$

For a load on section No. 3, $P_L = -M.D_2 + N.D_3.$

$$P_R = -G.D_3 + N.D_2.$$

For a load on section No. 4, $P_L = +M.D_2 - N.D_3.$

$$P_R = +G.D_3 - N.D_2.$$

For a load on section No. 5, $P_L = -M.D_2 + N.D_3.$

$$P_R = -G.D_3 + N.D_2.$$

For a load on section No. 6, $P_L = -N.D_4.$

$$P_R = +G.D_4.$$

For a load on section No. 7, $P_L = +N.D_4.$

$$P_R = -G.D_4.$$

In which $M = \dfrac{d_3 + d_4}{(d_1 + d_2)(d_3 + d_4) - d_{20}d_{30}}$,

$$G = \frac{d_1 + d_2}{(d_1 + d_2)(d_3 + d_4) - d_{20}d_{30}},$$

and

$$N = \frac{d_{20}}{(d_1 + d_2)(d_3 + d_4) - d_{20}d_{30}} = \frac{d_{30}}{(d_1 + d_2)(d_3 + d_4) - d_{20}d_{30}}.$$

The numerical values of the deflections should be used in the above formulæ without regard to their signs.

The resultant signs for the solved equations will give the character of stress in the rocker arms; + indicating tension, and − compression.

It will be noted that the values M, G, and N, are constants for the structure, so they can be used for any and every position of the moving load.

PROBLEMS

No. **52a.** Assume the previous structure to be symmetrical about a vertical line through the middle of Part II. Write the formulas for the stresses in the rocker arms for a load on each section.

No. **52b.** If sections 5, 6, and 7 be omitted from the previous structure it becomes similar to a double leaf bascule or trunnion drawbridge. Write the equations for the stress in P_L for a load on sections 2 and 3.

CHAPTER VII

MOVABLE BRIDGES

ART. 53. DRAWBRIDGES

THE name Drawbridge is applied to any structure which may be moved from its normal position in order to permit traffic to pass in a direction at right angles to its length, or to prevent the passage of traffic over it.

There are many types of drawbridges; the type to be selected at any given location depends on the requirements of the location. At many locations the requirements may be best met by a movable statically determinate structure. At most locations, however, some form of statically indeterminate structure is best

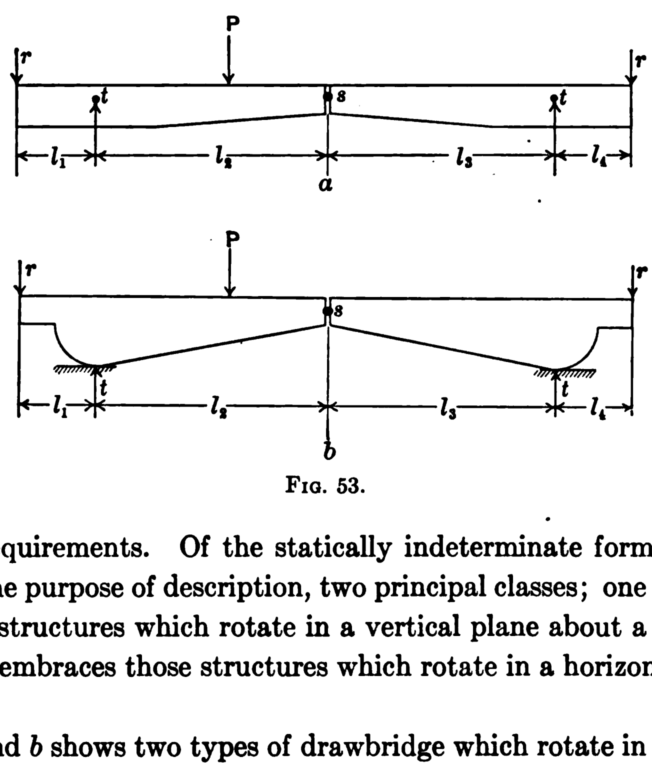


FIG. 53.

suited to the requirements. Of the statically indeterminate forms there may be said to be, for the purpose of description, two principal classes; one of which classes embraces those structures which rotate in a vertical plane about a horizontal axis. The other class embraces those structures which rotate in a horizontal plane about a vertical axis.

Fig. 53 *a* and *b* shows two types of drawbridge which rotate in a vertical plane about a horizontal axis.

In *a* the bridge is opened by rotating the two leaves about the fixed horizontal pins, or trunnions, *t*.

To permit the passage of live load a connection at s which will transmit shear, but not moment is made and supports are used at r which will exert either an upward or downward reaction as may be required as the system of loads passes over it.

In b the bridge is opened by rolling it backward on the horizontal supports at t, its other features are as described for a.

The leaves may be either a trussed framework or plate girders.

The deflections necessary to determine the unknown shear S_s, transmitted through the connection at s for either a or b of Fig. 53 are:

Δ_1 = the deflection at s for a load P on the left half for this half acting independently.

Δ_2 = the deflection at s for a load P on the right half for the half acting independently.

d_1 = the deflection at s for a vertical load of unity at s on the left half for this half acting independently.

d_2 = the deflection at s for a vertical load of unity at s on the right half for this half acting independently.

Then with the structure acting as a whole and a load P on the left half

$$S_s = \frac{\Delta_1}{d_1 + d_2},$$

and for a load on the right half

$$S_s = \frac{\Delta_2}{d_1 + d_2}.$$

If both halves are the same, $\Delta_1 = \Delta_2$ and $d_1 = d_2$, and omitting the subscripts for the deflections, then

$$S_s = \frac{\Delta}{2d}.$$

The drawbridges which rotate about a vertical axis are called Swing Bridges, and the best method of computing the necessary deflections for determining the unknown reactions for these structures will be shown in the next article..

PROBLEMS

No. **53a**. If the two leaves of Fig. 53a are 30-in. girder I-beams, having a moment of inertia of 9155 (inches to the fourth), $l_1 = l_4 = 8$ ft., $l_2 = l_3 = 24$ ft., construct the influence lines for shear and moment for a 8 ft. point to the left of the center.

No. **53b**. For the data of Problem 53a construct the influence line for moment and shear at the left support t.

No. **53c**. For the data of Problem 53a construct the influence line for reaction at the left support r.

ART. 54. REACTIONS FOR SWING BRIDGES BY MEANS OF THE RELATIONS BETWEEN CERTAIN DEFLECTIONS

Following the derivation of the Theorem of Three Moments in a former chapter of this book the application of the theorem to finding the reactions of swing bridges under the assumption that their moment of inertia was constant was shown. There is considerable uncertainty as to the exact amount of swing bridge reactions due to the manner of supporting such trusses in practice. For this reason many engineers object to using methods of great refinement in making the computations incident to their design, and consider that the assumption of a constant moment of inertia is sufficiently accurate. There is no better method of design at present for such structures than to make the first determination of the reactions and the sections of the members by using the theorem of three moments under the assumption of a constant moment of inertia of truss or girder; and then revise the sections by using the true reactions, in terms of the elastic deformation of the members, of the structure, as first determined.

In order to swing a drawbridge around a vertical axis it is necessary to remove certain of the supports, and to insure its stability during swinging it is necessary to provide a fixed point, on or about which the structure turns, and other supports at some distance from the fixed point to prevent tipping from any unbalanced load conditions. The determination of the nature of all these supports with reference to any specific location in order to meet the requirements as to time of operation, space in which to place the supports, etc., is a matter requiring a high degree of experience as well as theoretical knowledge. Repeated solution of the problem of swing bridge design has led to the evolution of two principal types, with reference to the nature of the center supports, the center bearing and the rim bearing bridges.

In the center bearing type the entire dead load of the structure is carried when open and during swinging on a center pivot. In order to lessen the friction between the upper and lower surfaces of the pivot, disks of some material with a small coefficient of friction or small conical rollers are interposed between these surfaces. This type has been successfully used for single-track railroad swing bridges up to four hundred feet in length. The general arrangement of the supports for this type is shown in Fig. 54*a*. In this figure the supports marked *e* and *c* must be removed before turning, the supports at *c* should be removed first. The swinging takes place on and about the *pivot*. The span is kept from tipping over by a sufficient number of trailing wheels *w*. The trailing wheels are slightly above the circular track, shown by the dotted line, and only come to a bearing when the structure is

tipped slightly. They are arranged in this manner to prevent the possibility of their taking any live or dead load reaction when the span is closed. The supports at c and e carry the full live load, and a part of the dead load which depends on the amount that the supported points are raised. Each truss or girder of this structure is supported at three points and in almost all cases the arms l_1 and l_2 are equal. For the case of equal arms the computation of the stresses in making the design may be greatly facilitated by the use of table No. 54a, which gives the amount of the reactions for a load at a great variety of positions based on the theorem of three moments for constant moment of inertia. With this table, influence lines, or tables, for stress in any member and moment or shear at any point may readily be constructed.

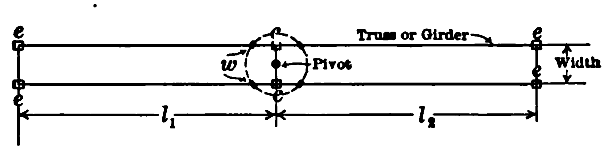


FIG. 54a.

The section of any member should be determined from a consideration of all possible simultaneous conditions of loading, which requires that where the ends of the span are not latched down four sets of stresses must be determined for the vertical loading.

First, for the dead load all carried on the center pivot.

Second, for an uplift at the ends of the draw.

Third, for live load on one arm and that arm acting as a simple span.

Fourth, for live load for the structure on three supports.

The first condition occurs when the structure is swinging or open.

The second is due to the fact that unless the ends of the bridge are latched down they must be lifted to prevent hammering under the passage of live load. The ends are generally lifted and the amount of lifting should be determined by the amount of negative reaction plus proper allowance for impact plus the requirement to overcome any unequal temperature effect; twice the maximum negative reaction as an uplift is ample for most cases to prevent hammering of the ends. The vertical deflection corresponding to a force equal to the greatest possible negative uplift is determined and each end raised half this distance.

The third condition is generally required by engineers in order to insure the stability of the structure under the passage of live load at a time when the end lifting devices might be out of order and not working.

The fourth condition is the one for which the structure is designed to act under

$R_1 = +P(1-k) - \frac{1}{4}P(k-k^3) \qquad R_2 = +Pk + \frac{1}{2}P(k-k^3) \qquad R_3 = -\frac{1}{4}P(k-k^3).$

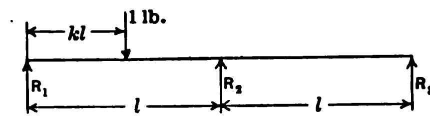


Table 54a

Reactions for a continuous beam of constant moment of inertia on three supports due to a load of unity spaced at various fractions of the length of one arm from the left end

k	R_1	R_2	R_3
$\frac{1}{11}$	+0.886	+0.136	−0.022
$\frac{2}{11}$	+0.774	+0.270	−0.044
$\frac{3}{11}$	+0.664	+0.399	−0.063
$\frac{4}{11}$	+0.557	+0.522	−0.079
$\frac{5}{11}$	+0.455	+0.635	−0.090
$\frac{6}{11}$	+0.359	+0.737	−0.096
$\frac{7}{11}$	+0.269	+0.826	−0.095
$\frac{8}{11}$	+0.187	+0.899	−0.086
$\frac{9}{11}$	+0.114	+0.954	−0.068
$\frac{10}{11}$	+0.051	+0.988	−0.039
$\frac{1}{12}$	+0.896	+0.124	−0.020
$\frac{2}{12}=\frac{1}{6}$	+0.793	+0.247	−0.040
$\frac{3}{12}=\frac{1}{4}$	+0.692	+0.367	−0.059
$\frac{4}{12}=\frac{2}{6}=\frac{1}{3}$	+0.593	+0.481	−0.074
$\frac{5}{12}$	+0.497	+0.589	−0.086
$\frac{6}{12}=\frac{3}{6}=\frac{1}{2}$	+0.406	+0.688	−0.094
$\frac{7}{12}$	+0.321	+0.775	−0.096
$\frac{8}{12}=\frac{4}{6}=\frac{2}{3}$	+0.241	+0.852	−0.093
$\frac{9}{12}=\frac{3}{4}$	+0.168	+0.914	−0.082
$\frac{10}{12}=\frac{5}{6}$	+0.103	+0.961	−0.064
$\frac{11}{12}$	+0.047	+0.990	−0.037
$\frac{1}{13}$	+0.904	+0.115	−0.019
$\frac{2}{13}$	+0.809	+0.220	−0.037
$\frac{3}{13}$	+0.715	+0.340	−0.055
$\frac{4}{13}$	+0.623	+0.447	−0.070
$\frac{5}{13}$	+0.534	+0.548	−0.082
$\frac{6}{13}$	+0.448	+0.643	−0.091
$\frac{7}{13}$	+0.366	+0.730	−0.096
$\frac{8}{13}$	+0.289	+0.807	−0.096
$\frac{9}{13}$	+0.218	+0.872	−0.090
$\frac{10}{13}$	+0.152	+0.926	−0.078
$\frac{11}{13}$	+0.094	+0.966	−0.060
$\frac{12}{13}$	+0.043	+0.991	−0.034
$\frac{1}{14}$	+0.911	+0.107	−0.018
$\frac{2}{14}=\frac{1}{7}$	+0.822	+0.213	−0.035
$\frac{3}{14}$	+0.735	+0.316	−0.051

k	R_1	R_2	R_3
$\frac{4}{14}=\frac{2}{7}$	+0.649	+0.417	−0.066
$\frac{5}{14}$	+0.565	+0.513	−0.078
$\frac{6}{14}=\frac{3}{7}$	+0.484	+0.603	−0.087
$\frac{7}{14}=\frac{1}{2}$	+0.406	+0.688	−0.094
$\frac{8}{14}=\frac{4}{7}$	+0.332	+0.764	−0.096
$\frac{9}{14}$	+0.263	+0.831	−0.094
$\frac{10}{14}=\frac{5}{7}$	+0.198	+0.889	−0.087
$\frac{11}{14}$	+0.139	+0.936	−0.075
$\frac{12}{14}=\frac{6}{7}$	+0.086	+0.971	−0.057
$\frac{13}{14}$	+0.039	+0.993	−0.032
$\frac{1}{15}$	+0.917	+0.100	−0.017
$\frac{2}{15}$	+0.834	+0.199	−0.033
$\frac{3}{15}=\frac{1}{5}$	+0.752	+0.296	−0.048
$\frac{4}{15}$	+0.671	+0.391	−0.062
$\frac{5}{15}=\frac{1}{3}$	+0.593	+0.481	−0.074
$\frac{6}{15}=\frac{2}{5}$	+0.516	+0.568	−0.084
$\frac{7}{15}$	+0.442	+0.649	−0.091
$\frac{8}{15}$	+0.371	+0.724	−0.095
$\frac{9}{15}=\frac{3}{5}$	+0.304	+0.792	−0.096
$\frac{10}{15}=\frac{2}{3}$	+0.241	+0.852	−0.093
$\frac{11}{15}$	+0.182	+0.903	−0.085
$\frac{12}{15}=\frac{4}{5}$	+0.128	+0.944	−0.072
$\frac{13}{15}$	+0.079	+0.975	−0.054
$\frac{14}{15}$	+0.037	+0.993	−0.030
$\frac{1}{16}$	+0.922	+0.094	−0.016
$\frac{2}{16}=\frac{1}{8}$	+0.844	+0.187	−0.031
$\frac{3}{16}$	+0.767	+0.278	−0.045
$\frac{4}{16}=\frac{2}{8}=\frac{1}{4}$	+0.692	+0.367	−0.059
$\frac{5}{16}$	+0.617	+0.454	−0.071
$\frac{6}{16}=\frac{3}{8}$	+0.544	+0.537	−0.081
$\frac{7}{16}$	+0.474	+0.614	−0.088
$\frac{8}{16}=\frac{4}{8}=\frac{1}{2}$	+0.406	+0.688	−0.094
$\frac{9}{16}$	+0.341	+0.755	−0.096
$\frac{10}{16}=\frac{5}{8}$	+0.280	+0.815	−0.095
$\frac{11}{16}$	+0.222	+0.869	−0.091
$\frac{12}{16}=\frac{6}{8}=\frac{3}{4}$	+0.168	+0.914	−0.082

k	R_1	R_2	R_3
$\frac{13}{16}$	+0.119	+0.950	−0.069
$\frac{14}{16}=\frac{7}{8}$	+0.074	+0.977	−0.051
$\frac{15}{16}$	+0.034	+0.994	−0.028
$\frac{1}{17}$	+0.927	+0.088	−0.015
$\frac{2}{17}$	+0.853	+0.176	−0.029
$\frac{3}{17}$	+0.781	+0.262	−0.043
$\frac{4}{17}$	+0.709	+0.347	−0.056
$\frac{5}{17}$	+0.639	+0.428	−0.067
$\frac{6}{17}$	+0.570	+0.507	−0.077
$\frac{7}{17}$	+0.503	+0.583	−0.086
$\frac{8}{17}$	+0.438	+0.654	−0.092
$\frac{9}{17}$	+0.375	+0.720	−0.095
$\frac{10}{17}$	+0.316	+0.780	−0.096
$\frac{11}{17}$	+0.259	+0.835	−0.094
$\frac{12}{17}$	+0.206	+0.883	−0.089
$\frac{13}{17}$	+0.156	+0.923	−0.079
$\frac{14}{17}$	+0.110	+0.956	−0.066
$\frac{15}{17}$	+0.069	+0.980	−0.049
$\frac{16}{17}$	+0.032	+0.995	−0.027
$\frac{1}{18}$	+0.931	+0.083	−0.014
$\frac{2}{18}=\frac{1}{9}$	+0.861	+0.166	−0.027
$\frac{3}{18}=\frac{1}{6}$	+0.793	+0.248	−0.041
$\frac{4}{18}=\frac{2}{9}$	+0.725	+0.328	−0.053
$\frac{5}{18}$	+0.658	+0.406	−0.064
$\frac{6}{18}=\frac{3}{9}=\frac{1}{3}$	+0.593	+0.481	−0.074
$\frac{7}{18}$	+0.529	+0.554	−0.083
$\frac{8}{18}=\frac{4}{9}$	+0.467	+0.622	−0.089
$\frac{9}{18}=\frac{3}{6}=\frac{1}{2}$	+0.406	+0.688	−0.094
$\frac{10}{18}=\frac{5}{9}$	+0.348	+0.748	−0.096
$\frac{11}{18}$	+0.293	+0.803	−0.096
$\frac{12}{18}=\frac{6}{9}=\frac{2}{3}$	+0.241	+0.852	−0.093
$\frac{13}{18}$	+0.191	+0.895	−0.086
$\frac{14}{18}=\frac{7}{9}$	+0.145	+0.932	−0.077
$\frac{15}{18}=\frac{5}{6}$	+0.103	+0.961	−0.064
$\frac{16}{18}=\frac{8}{9}$	+0.065	+0.982	−0.047
$\frac{17}{18}$	+0.030	+0.996	−0.026

k	R_1	R_2	R_3
$\frac{1}{19}$	+0.934	+0.079	−0.013
$\frac{2}{19}$	+0.869	+0.157	−0.026
$\frac{3}{19}$	+0.804	+0.235	−0.039
$\frac{4}{19}$	+0.739	+0.311	−0.050
$\frac{5}{19}$	+0.676	+0.386	−0.062
$\frac{6}{19}$	+0.613	+0.458	−0.071
$\frac{7}{19}$	+0.552	+0.528	−0.080
$\frac{8}{19}$	+0.493	+0.594	−0.087
$\frac{9}{19}$	+0.434	+0.658	−0.092
$\frac{10}{19}$	+0.378	+0.717	−0.095
$\frac{11}{19}$	+0.325	+0.771	−0.096
$\frac{12}{19}$	+0.274	+0.821	−0.095
$\frac{13}{19}$	+0.225	+0.866	−0.091
$\frac{14}{19}$	+0.179	+0.905	−0.084
$\frac{15}{19}$	+0.136	+0.938	−0.074
$\frac{16}{19}$	+0.097	+0.965	−0.062
$\frac{17}{19}$	+0.061	+0.984	−0.045
$\frac{18}{19}$	+0.028	+0.996	−0.024
$\frac{1}{20}$	+0.938	+0.075	−0.013
$\frac{2}{20}=\frac{1}{10}$	+0.875	+0.150	−0.025
$\frac{3}{20}$	+0.814	+0.223	−0.037
$\frac{4}{20}=\frac{2}{10}=\frac{1}{5}$	+0.752	+0.296	−0.048
$\frac{5}{20}=\frac{1}{4}$	+0.692	+0.367	−0.059
$\frac{6}{20}=\frac{3}{10}$	+0.632	+0.436	−0.068
$\frac{7}{20}$	+0.573	+0.504	−0.077
$\frac{8}{20}=\frac{4}{10}=\frac{2}{5}$	+0.516	+0.568	−0.084
$\frac{9}{20}$	+0.460	+0.630	−0.090
$\frac{10}{20}=\frac{5}{10}=\frac{1}{2}$	+0.406	+0.688	−0.094
$\frac{11}{20}$	+0.354	+0.742	−0.096
$\frac{12}{20}=\frac{6}{10}=\frac{3}{5}$	+0.304	+0.792	−0.096
$\frac{13}{20}$	+0.256	+0.838	−0.094
$\frac{14}{20}=\frac{7}{10}$	+0.211	+0.878	−0.089
$\frac{15}{20}=\frac{3}{4}$	+0.168	+0.914	−0.082
$\frac{16}{20}=\frac{8}{10}=\frac{4}{5}$	+0.128	+0.944	−0.072
$\frac{17}{20}$	+0.091	+0.968	−0.059
$\frac{18}{20}=\frac{9}{10}$	+0.057	+0.986	−0.043
$\frac{19}{20}$	+0.027	+0.996	−0.023

passage of live load and is the governing one for most of the members near the center when taken with the first and second simultaneous conditions.

Fig. 54*b* shows the outline, and Table No. 54*b* gives the sections of the members of a truss for a swing span which has been designed for the four conditions of loading just described; the stresses for the fourth condition of loading used in making the design were based on reactions which were determined by the use of the theorem of three moments for constant moment of inertia. The calculation of the true reactions of the truss for the cross-sections of the members as thus determined is also given in Table No. 54*b*. The reaction R_1 is determined from the relation $R_1 = \frac{D}{d}$, in which D is deflection at a due to a load of unity at any point, and d the deflection at a due to a load of unity at a of the left arm when the support at the left end of the arm is considered removed. Having computed R_1 in this manner, the reactions R_2 and R_3 are easily computed from the laws of static equilibrium.

Special attention perhaps should be called to the manner of determining the portion of D and d produced by the changes in length of the members of the right half of the draw.

It will be noted that for a load of 1 pound at a, b, c, d or e, the negative reaction at the right end of the right arm is 1 pound, 0.8, 0.6, 0.4, or 0.2 of a pound respectively, and, therefore, if 48.837 is the part of D produced by the members of the right arm for 1 pound at a that:

The part of D due to the right arm for 1 pound at $b = 48.837 \times .8 = 34.070$.
The part of D due to the right arm for 1 pound at $c = 48.837 \times .6 = 29.302$.
The part of D due to the right arm for 1 pound at $d = 48.837 \times .4 = 19.535$.
The part of D due to the right arm for 1 pound at $e = 48.837 \times .2 = \ \ 9.767$.

The true values of the reactions for unit loads at the panel points for the truss with members as stated are shown, for comparison, with the same reactions determined from the theorem of three moments and constant moment of inertia. With the more exact values of the reactions now computed a new determination of the stresses for the fourth condition of loading should be made, and new cross-sections determined for the various members.

Fig. 54*c* shows the outline and general dimensions of the truss of Arts. 41 and 42, and Table No. 54*c* shows the cross-sections for the members of a truss which was designed to be carried on two supports. Table No. 54*c* also gives the computations for the reactions due to unit loads at the various panel points for this truss supported at three points and the values of the same reaction if derived from the theorem of three moments and constant moment of inertia.

TABLE NO. 54b

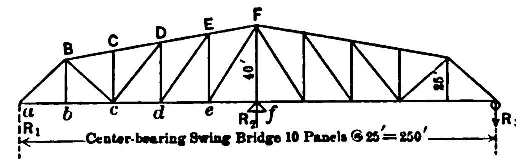


FIG. 54b.

Member.	Length.	Area.	L/A	T_a	T_b	T_c	T_d	T_e	LT_a/A	LT_a^2/A	LT_aT_b/A	LT_aT_c/A	TL_aT_d/A	LT_aT_e/A
BD	50.56	27.78	1.820	+1.758	+0.879				+3.200	5.626	2.813			
DE	25.28	27.78	0.910	+2.333	+1.556	+0.778			+2.124	4.956	3.305	1.653		
EF	25.28	18.90	1.338	+2.790	+2.092	+1.395	+0.697		+3.733	10.415	7.810	5.208	2.602	
ac	50.00	24.82	2.014	−1.000					−2.014	2.014				
cd	25.00	24.82	1.007	−2.307	−1.538	−0.769			−2.323	5.360	3.579	1.786		
de	25.00	31.12	0.803	−2.759	−2.069	−1.379	−0.690		−2.216	6.114	4.585	3.056	1.528	
ef	25.00	44.97	0.556	−3.125	−2.500	−1.875	−1.250	−0.625	−1.737	5.429	4.343	3.258	2.171	1.086
aB	35.35	32.87	1.076	+1.414					+1.521	2.151				
Bc	35.35	39.20	0.902	−1.045	−1.229				−0.943	0.985	1.159			
cD	41.03	17.85	2.299	+0.933	+1.097	+1.262			+2.145	2.001	2.353	2.707		
Dd	32.50	24.61	1.321	−0.654	−0.769	−0.885			−0.864	0.565	0.664	0.764		
dE	44.03	22.05	1.997	+0.794	+0.934	+1.074	+1.215		+1.586	1.260	1.482	1.703	1.927	
Ee	36.25	37.07	0.978	−0.586	−0.690	−0.793	−0.897		−0.573	0.336	0.395	0.455	0.514	
eF	47.17	28.00	1.685	+0.691	+0.813	+0.935	+1.058	+1.179	+1.165	0.805	0.947	1.089	1.233	1.374
Ff/2	20.00	91.48	0.217	−2.000	−1.800	−1.600	−1.400	−1.200	−0.435	0.870	0.783	0.696	0.609	0.522
						Sum for left arm			=	48.837	34.218	22.375	10.584	2.982
						Sum for right arm			=	48.837	39.070	29.302	19.535	9.767
						Total			=	97.674	73.288	51.677	30.119	12.749

For 1 lb. at	R_1	R_2	R_3	For Beam of Constant Moment of Inertia.		
				$R_1 = +P(1-k) - \frac{1}{4}(P(k-k^3)$	$R_2 = +Pk + \frac{1}{2}P(k-k^3)$	$R_3 = -\frac{1}{4}P(k-k^3)$
b	$\frac{73.288}{97.674} = +0.750$	+0.300	−0.050	+0.752	+0.296	−0.048
c	$\frac{51.677}{97.674} = +0.529$	+0.542	−0.071	+0.516	+0.568	−0.084
d	$\frac{30.119}{97.674} = +0.308$	+0.784	−0.092	+0.304	+0.792	−0.096
e	$\frac{12.749}{97.674} = +0.131$	+0.938	−0.069	+0.128	+0.944	−0.072

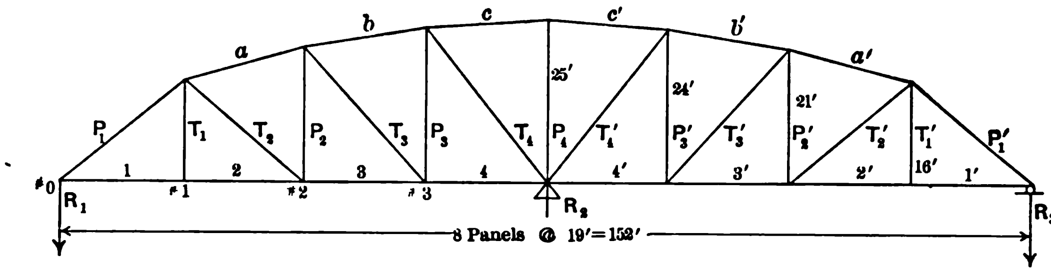


FIG. 54c

TABLE 54c

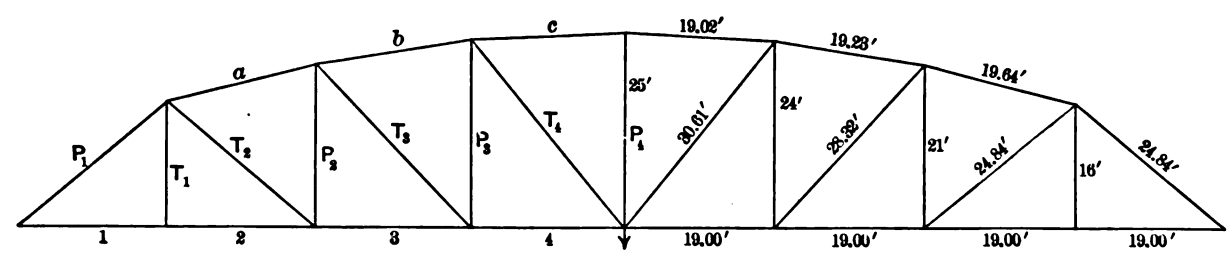
Members	L'gth in Feet L	Area in Sq. Ins. A	L/A	1 lb. at 0 (↓) T_0	1 lb. at 1 (↓) T_1	1 lb. at 2 (↓) T_2	1 lb. at 3 (↓) T_3	$\frac{LT_0}{A}$	$\frac{LT_0T_0}{A}$	$\frac{LT_0T_1}{A}$	$\frac{LT_0T_2}{A}$	$\frac{LT_0T_3}{A}$
P_1	24.84	16.72	1.49	+1.552	0	0	0	+2.313	+ 3.589	0	0	0
P_2	21.00	5.29	3.97	+0.524	+0.762	+1.000	0	+2.080	+ 1.090	+ 1.585	+ 2.080	0
P_3	24.00	5.29	4.54	+0.625	+0.750	+0.875	+1.000	+2.837	+ 1.773	+ 2.128	+ 2.483	+ 2.837
$P_4/2$	25.00	5.29	4.72	+0.320	+0.240	+0.160	+0.080	+1.510	+ 0.242	+ 0.181	+ 0.121	+ 0.060
T_1	16.00	5.29	3.03	0	+1.000	0	0	0	0	0	0	0
T_2	24.84	3.75	6.62	−0.814	−1.183	0	0	−5.388	+ 4.385	+ 6.374	0	0
T_3	28.32	7.06	4.01	−0.843	−1.011	−1.180	0	−3.380	+ 2.850	+ 3.417	+ 3.989	0
T_4	30.61	7.06	4.33	−1.071	−1.122	−1.173	−1.224	−4.638	+ 4.967	+ 5.204	+ 5.440	+ 5.677
a	19.65	13.89	1.41	+1.871	+0.936	0	0	+2.638	+ 4.936	+ 2.470	0	0
b	19.24	13.89	1.38	+2.405	+1.603	+0.802	0	+3.319	+ 7.982	+ 5.320	+ 2.662	0
c	19.03	13.89	1.37	+3.044	+2.283	+1.522	+0.761	+4.171	+ 12.694	+ 9.521	+ 6.348	+ 3.174
1	19.00	4.50	4.22	−1.187	0	0	0	−5.010	+ 5.946	0	0	0
2	19.00	4.50	4.22	−1.187	0	0	0	−5.010	+ 5.946	0	0	0
3	19.00	8.75	2.17	−1.809	−0.905	0	0	−3.926	+ 7.102	+ 3.553	0	0
4	19.00	9.37	2.03	−2.375	−1.583	−0.792	0	−4.822	+ 11.452	+ 7.633	+ 3.819	0
								Left half . . Σ =	+ 74.954	+ 47.386	+26.942	+11.748
								Right half . Σ =	+ 74.954	+ 56.216	+37.477	+18.739
								Total . . . . . Σ =	+149.908	+103.602	+64.419	+30.487

		1 lb. at 1	1 lb. at 2	1 lb. at 3
Exact Method	R_1	+0.691	+0.430	+0.203
	R_2	+0.368	+0.638	+0.843
	R_3	−0.059	−0.068	−0.047
Constant Moment of Inertia	R_1	+0.691	+0.406	+0.168
	R_2	+0.367	+0.687	+0.914
	R_3	−0.059	−0.094	−0.082

$$100{,}000 \text{ lbs. at } 1,\ R_1 = \frac{103.602}{149.908} \times 100{,}000 = 69{,}120 \text{ lbs.}$$

$$100{,}000 \text{ lbs. at } 2,\ R_1 = \frac{64.419}{149.908} \times 100{,}000 = 42{,}970 \text{ lbs.}$$

$$100{,}000 \text{ lbs. at } 3,\ R_1 = \frac{30.487}{149.908} \times 100{,}000 = 20{,}340 \text{ lbs.}$$



Load of Unity

FIG. 54*d*.

TABLE No. 54*d*

	L	A	$\frac{L}{A}$	Stress due to Unity,	Elongation due to Unity.	Deflection Increments.
P_1	24.84	16.72	1.49	− .776	$\frac{1.49 \times .776}{E}$	+ .897 $\div E$
P_2	21.00	5.29	3.97	− .262	$\frac{3.97 \times .262}{E}$	+ .273 $\div E$
P_3	24.00	5.29	4.54	− .312	$\frac{4.54 \times .312}{E}$	+ .442 $\div E$
P_4	25.00	5.29	4.72	+ .160	$\frac{4.72 \times .160}{E} \div 2$	+ .060 $\div E$
T_1	16.00	5.29	3.03	0	$\frac{3.03 \times 0}{E}$	
T_2	24.84	3.75	6.62	+ .407	$\frac{6.62 \times .407}{E}$	+ 1.097 $\div E$
T_3	28.32	7.06	4.01	+ .420	$\frac{4.01 \times .420}{E}$	+ .707 $\div E$
T_4	30.61	7.06	4.33	+ .536	$\frac{4.33 \times .536}{E}$	+ 1.2430 $\div E$
a	19.64	13.89	1.41	− .935	$\frac{1.41 \times .935}{E}$	+ 1.2330 $\div E$
b	19.23	13.89	1.38	−1.199	$\frac{1.38 \times 1.199}{E}$	+ 1.9840 $\div E$
c	19.02	13.89	1.37	−1.522	$\frac{1.37 \times 1.522}{E}$	+ 3.1720 $\div E$
1	19.00	4.50	4.22	+ .593	$\frac{4.22 \times .593}{E}$	+ 1.4840 $\div E$
2	19.00	4.50	4.22	+ .593	$\frac{4.22 \times .593}{E}$	+ 1.4840 $\div E$
3	19.00	8.75	2.17	+ .905	$\frac{2.17 \times .905}{E}$	+ 1.7770 $\div E$
4	19.00	9.37	2.03	+1.187	$\frac{2.03 \times 1.187}{E}$	+ 2.8610 $\div E$
						18.714 $\div E$

$$\frac{.1450 \times E}{2 \times 18.714} = R = 116{,}200 \text{ lbs.}$$

Computation of the true reactions for the trusses of both Fig. 54*b* and 54*c* which are very different in the purpose for which they were designed shows that more of the loads are carried to the center support than for a structure of constant moment of inertia. The values of the reactions as determined by the two methods should be compared by the student with a view to judging the accuracy of the three moment method.

In Art. 42 the reaction exerted by the center support for the truss of Fig. 54*d* when this truss is supported at three points, was determined by means of Castigliano's Second Theorem; Tables No. 44*c* and 54*d* enable the reaction to be determined in a very simple manner, as is indicated below Table No. 54*d*.

PROBLEMS

No. **54***a*. By means of the table of true reactions for the truss of Fig. 54*b* construct the influence ine for the stress in each of the members *Bc*, *cF*, *ef*, and *BC* for the truss acting as a whole on three supports.

No. **54***b*. By means of the influence lines obtained in solving Problem 54*a* determine the maximum stresses of tension and compression in the members *Bc*, *eF*, *ef*, and *BC* for Cooper's E_{50} loading. One-half the loading to be carried by one truss.

No. **54***c*. Determine the amount of uniform load per linear foot of truss which would produce the same maximum stresses obtained in solving Problem 54*b*.

No. **54***d*. With the results for Problem 54*c* before you is it possible to represent the effect of the train loading in producing the stresses of Problem 54*b* by a uniform load ?

ART. 55. REACTIONS FOR SWING BRIDGES BY MEANS OF THE RELATION BETWEEN CERTAIN DEFLECTIONS

For swing bridges which are of great length or width, the center-bearing type does not furnish enough stability during turning, and further, the construction of a center cross girder of sufficient strength to carry all the dead load to the center pivot becomes practically impossible. In order to distribute the loads over the center pier and to carry them to the pier in as direct a manner as possible the rim-bearing type has been developed. In this type the dead load of the bridge when swinging and open and a part of the dead and live loads of the bridge when closed, is transferred through a circular girder supported on conical wheels to the pier. The circular girder is held rigidly against lateral displacement to a fixed point at its center. The circular girder should be loaded in a symmetrical manner, and this can best be done by making the center panel of the truss equal to the width of the bridge center to center of the trusses. Fig. 55*a* shows the points of support for the superstructure of a rim-bearing swing bridge. The supports marked *e* are movable

and must be removed before the bridge is opened. The loads brought by the main trusses to the points c are distributed over the center pier in many ways. For bridges where the points c, c, c and c are at the corners of a square the arrangement of the equalizing and distributing girders is quite simple. For swing bridges of great width and having more than two trusses the design of an efficient and truly

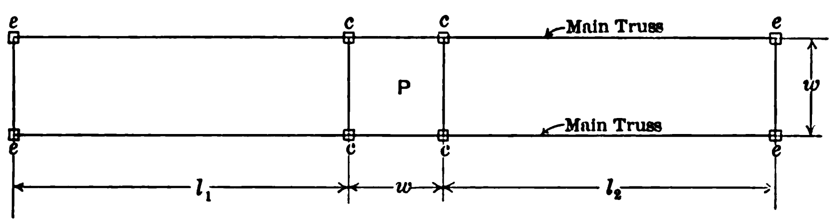


FIG. 55a.

distributing system of girders requires much originality and ability in designing. The bridges in New York City which carry the city streets over the Harlem River afford excellent examples of the manner in which this difficult problem has been met by many different designers. With the load of the structure brought to four points over the center pier and these points forming the corners of a square the following six arrangements are common forms of construction.

In the first, shown by Fig. 55b, the loaded points c are placed directly on the drum D. The loads are carried to the pier through the drum and the conical wheels below it, with the load applied to the drum at only four points the torsional stresses in it are very high, and the wheels which are under the points c must carry most of the load, as no matter how heavy and deep the drum may be made it will still be flexible and therefore does not distribute the loads.

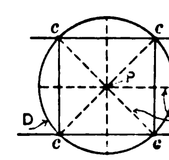


FIG. 55b.

The radial struts S shown by the dotted lines do not carry load, but are used to prevent lateral displacement of the drum and to hold it vertical and true to its circular form. The drum and center pier may be kept of small diameter by this method of loading. The weight of material required is also small as the loads are not carried through long paths.

In the second arrangement, shown in Fig. 55c, part of the load is brought to the center pivot P by means of the four radial girders. This makes a larger drum necessary than for the first case, and it is more difficult to secure good distribution through the longer segments of the drum. The part of the load brought to the center pivot offers less resistance to turning than if applied to the drum, due to the much smaller lever arm of the friction and aids in securing good distribution on the masonry of the center pier.

In the third and fourth arrangements, which are shown in Fig. 55d and Fig. 55e respectively, the number of the loaded point of the drum has been increased by means of the girders V to eight, thereby rendering a much better distribution of the load over the wheels carrying the drum and permitting a much larger structure to be carried with safety. In the fifth arrangement shown in Fig. 55f the load is applied to eight points of the drum, and as no auxiliary girders are required, where there is enough depth the arrangement is a good one, as the drum may be made very small in diameter. In the sixth arrangement, which is shown in Fig. 55g, the number of loaded points is increased to sixteen, thereby giving still better distribution of the load over the wheels carrying the drum and making it the best arrangement for large and important bridges. In distributing the loads from the points c to the masonry care should be taken in cases where these points do not come at the corner of a square to use girders of proper stiffness to distribute the loads uni-

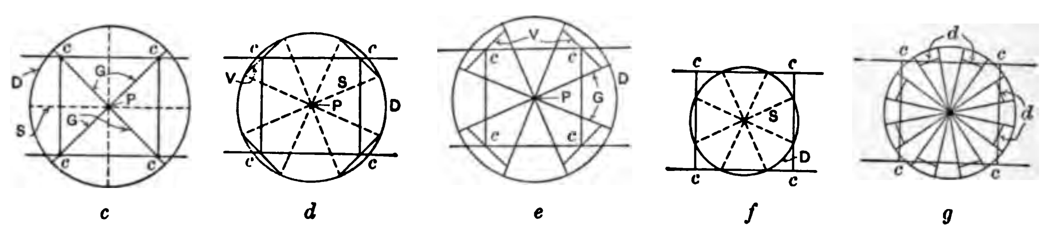


FIG. 55.

formly. Even in the simple case of Fig. 55g the longitudinal and transverse girders which transfer the loads from points c to point d must be of equal stiffness as well as equal strength for proper distribution, and sometimes as there is not the same space available for both longitudinal and transverse girders there must be extra material, in one to make it have the same stiffness as the other.

The points c of Fig. 55a are the points to which the reactions R_2 and R_3 of Fig. 55h are applied; in order that both these reactions be upward for live load on one arm of the draw and therefore as small as possible, no diagonals should be used in the center panel. Any negative live load reaction at either R_2 or R_3 in excess of the positive dead-load reactions would be very objectionable, as it would only be possible to obtain it by means of some mechanical device which would permit of ready disconnection for opening the draw. As the drum, distributing girder and floor of such a swing bridge have enough excess of strength to take any shear across the center panel, due to any unbalanced load which can occur during turning, diagonals in the center panel are not necessary.

Trusses of the form of that of Fig. 55h have been found from actual observation and from theoretical study of their elastic deformations to be very similar in

Table No. 55h

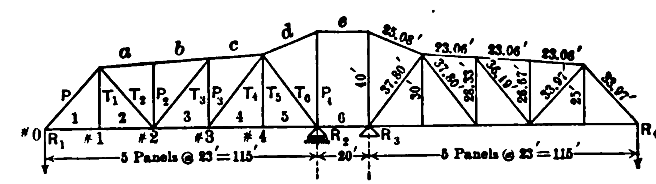


Fig. 55h

Members	Length in Feet L	Area in Sq. Ins. A	L/A	T_0	T_1	T_2	T_3	T_4	$\frac{T_0L}{A}$	$\frac{T_0T_0L}{A}$	$\frac{T_0T_1L}{A}$	$\frac{T_0T_2L}{A}$	$\frac{T_0T_3L}{A}$	$\frac{T_0T_4L}{A}$
P_1	33.97	28.31	1.200	+1.359	0	0	0	0	+1.631	+ 2.217	0	0	0	0
P_2	26.67			0	0	0	0	0	0	0	0	0	0	0
P_3	28.33	19.71	1.437	−0.823	−0.882	−0.941	0	0	−1.182	+ 0.937	+ 1.042	+ 1.112	0	0
P_4	40.00	18.31	2.184	−1.250	−1.000	−0.750	−0.500	−0.250	−2.730	+ 3.412	+ 2.730	+ 2.047	+ 1.365	+ 0.682
T_1	25.00			0	+1.000	0	0	0	0	0	0	0	0	0
T_2	33.97	26.80	1.267	−1.190	−1.274	0	0	0	−1.508	+ 1.795	+ 1.921	0	0	0
T_3	36.49	22.32	1.635	+1.128	+1.208	+1.288	0	0	+1.844	+ 2.080	+ 2.228	+ 2.375	0	0
T_4	37.80	19.95	1.895	+1.037	+1.111	+1.186	+1.260	0	+1.965	+ 2.038	+ 2.183	+ 2.330	+ 2.476	0
T_5	30.00			0	0	0	0	+1.000	0	0	0	0	0	0
T_6	37.80	37.10	1.019	+0.315	±0.000	−0.315	−0.630	−0.945	+0.321	+ 0.101	± 0.000	− 0.101	− 0.202	− 0.303
a	23.06	28.31	0.815	+1.728	+0.864	0	0	0	+1.408	+ 2.33	+ 1.216	0	0	0
b	23.06	28.31	0.815	+1.728	+0.864	0	0	0	+1.408	+ 2.43	+ 1.217	0	0	0
c	23.06	28.31	0.815	+2.442	+1.628	+0.814	0	0	+1.990	+ 4.860	+ 3.240	+ 1.620	0	0
d	25.08	26.95	0.931	+3.135	+2.508	+1.881	+1.254	+0.627	+2.919	+ 9.150	+ 7.320	+ 5.490	+ 3.660	+ 1.830
$e/2$	10.00	25.73	0.389	+2.875	+2.300	+1.725	+1.150	0.575	+1.118	+ 3.214	+ 2.571	+ 1.929	+ 1.286	+ 0.643
1	23.00	19.71	1.167	−0.920	0	0	0	0	−1.074	+ 0.988	0	0	0	0
2	23.00	19.71	1.167	−0.920	0	0	0	0	−1.074	+ 0.988	0	0	0	0
3	23.00	28.36	0.811	−2.436	−1.624	−0.812	0	0	−1.976	+ 4.814	+ 3.210	+ 1.605	0	0
4	23.00	28.36	0.811	−3.067	−2.300	−1.533	−0.767	0	−2.487	+ 7.628	+ 5.721	+ 3.814	+ 1.907	0
5	23.00	28.36	0.811	−3.067	−2.300	−1.533	−0.767	0	−2.487	+ 7.628	+ 5.721	+ 3.814	+ 1.907	0
6/2	10.00	31.42	0.318	−2.875	−2.300	−1.725	−1.150	−0.575	−0.914	+ 2.628	+ 2.102	+ 1.577	+ 1.051	+ 0.526
									Left half Σ =	+ 59.380	+42.422	+27.612	+13.450	+ 3.378
									Right half Σ =	+ 59.380	+47.504	+35.628	+23.752	+11.876
									Total Σ =	+118.760	+89.926	+63.240	+37.202	+15.254

For load of 1 lb. at 1, $R_1 = \frac{+\ 89.926}{+118.760} = +0.757$

For load of 1 lb. at 2, $R_1 = \frac{+\ 63.240}{+118.760} = +0.532$

For load of 1 lb. at 3, $R_1 = \frac{+\ 37.202}{+118.760} = +0.313$

For load of 1 lb. at 4, $R_1 = \frac{+\ 15.254}{+118.760} = +0.128$

True Reactions for Sections as stated in the Table No. 55h

Load at	R_1	R_2	R_3	R_4
No. 1	+0.757	+0.243	+0.043	−0.043
No. 2	+0.532	+0.468	+0.068	−0.068
No. 3	+0.313	+0.687	+0.087	−0.087
No. 4	+0.128	+0.872	+0.072	−0.072

Reactions for Beam of Constant Moment of Inertia

Load at	R_1	R_2	R_3	R_4
No. 1	+0.752	+0.248	+0.048	−0.048
No. 2	+0.516	+0.484	+0.084	−0.084
No. 3	+0.304	+0.696	+0.096	−0.096
No. 4	+0.128	+0.872	+0.072	−0.072

their action to that of a continuous girder consisting of the two end spans; that is to the structure formed by omitting the center span. The method of design for such trusses is to use the table of reactions, No. 54*a*, in determining the stresses, just as was done for the center bearing swing bridges of two equal arms. It may be noted that

R_1 for this case will equal R_1 of Table No. **54*a***.

R_4 for this case will equal R_3 of Table No. **54*a***.

R_2 for this case will equal $R_1 - P$.

R_3 for this case will equal R_4, but with opposite sign.

The areas of the members of the truss of Fig. 55*h*, as shown in the third column of Table 55*a*, were determined from the four conditions of loading as outlined in the previous article. With these areas the true reactions for unit loads should be computed as indicated in the balance of the table, and with these reactions true stresses and final areas for the members found. The reactions for two equal spans and constant moment of inertia are recorded for comparison with the true reactions. In general for swing bridges it may be said that for purposes of estimate and all preliminary work, the theorem of three moments for beams with constant moment of inertia gives values of the reactions which are close enough to be used with all confidence. For structures of importance which are to be built the exact method of computing the reactions should be used.

For a very valuable paper on the effect of the form of trusses in influencing their reactions and stresses, see Proc. of Engrs. Soc. of Western Pennsylvania, February, 1900, by Willis Whited, Asst. Engineer, Dept. of Public Works, Pittsburgh, Pa.

PROBLEMS

No. **55*a***. By means of the table of true reactions for the truss of Fig. 55*h* construct the influence line for the stress in each of the members T_2, T_6, T_4, and a, for the truss acting as a whole.

No. **55*b***. By means of the influence lines obtained in solving Problem **55*a*** determine the maximum stresses of tension and compression in the members T_2, T_6, P_4, and a, for Cooper's E_{50} loading.

CHAPTER VIII

THE ARCH WITH AN OPEN FRAME-WORK RIB

ART. 56. INTRODUCTORY

THE proper form for an arch which will be most economical in cost is very difficult of determination. The design of any bridge or structure should always conform to the local conditions to a considerable degree. The conditions justifying the use of an arch bridge would be the demand for an ornamental structure and the surety of securing unyielding foundations. If, in addition, the location of the roadway surface on which traffic must be carried is at a considerable height above the lowest possible point of the superstructure, and considerable clear width and height

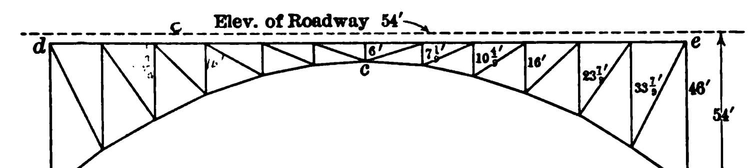


FIG. 56*a*.

is required for the needs of traffic or navigation below the superstructure, the conditions clearly demand an arch of some form. It is no part of the purpose of this chapter to attempt to indicate the type of arch rib to be employed at any given location. One of the most common types of open framework arch rib is known as the Spandrel-Braced Arch Rib. To illustrate the methods and computations which follow, the demands of the supposed location will be considered best met by a series of such arches of 200 ft. span and a maximum rise of bottom chord of 40 ft.

The outline and dimensions for the arched truss are as shown in Fig. 56*a*, the bottom chord points of which truss have been made to lie in a parabola.

Such arches are generally made with either two or three hinges and are then known as the two-hinged arch and the three-hinged arch respectively.

If the truss of Fig. 56*a* is supported at *a* by a fixed pin about which the structure may freely rotate and at *b* by a pin which is free to move in the direction

of the length of the truss, under the elastic deformations of its members or from any cause, it is a simple truss. That is, as can readily be seen, all its stresses may be readily computed by the methods of statics.

This form of truss is rarely built, however, as its depth is least where the bending moment is greatest, and it is therefore very uneconomical. It would also be undesirable for the reason that its deflection would be excessive. If the truss of Fig. 56*a* be supported at *a* and *b* by fixed pins about which the structure may freely rotate and at the same time a pin be introduced at *c*, and one of the center top chord sections be removed, or what amounts to the same thing, this chord section be given a connection at one end which permits of free expansion in the direction of its length, thus making it unable to carry direct stress, the structure becomes a three-hinged arch, the stresses of which may be computed by statics.

The location of the pin *c* and the superfluous chord are in the proper place, as shown in Fig. 56*b*, for economy of cost.

With this arrangement the truss is symmetrical with reference to its center line, except for the expansion arrangement in the superfluous chord.

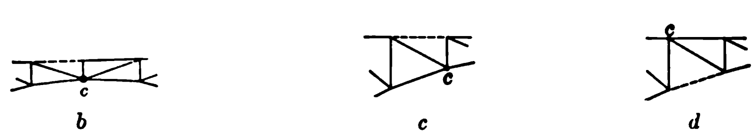


FIG. 56.

Having the pin *c* in the bottom chord rather than above it gives the bottom chord an only slightly and almost uniformly varying section from the center to the end of span. This symmetry and absence of sudden change of section of the main member of the arch unquestionably lead to economy and excellence of design.

The third pin *c* and the superfluous chord may be located almost anywhere between *a* and *b*, for instance, as shown in Figs. 56*c* and 56*d*, but any location other than that shown in Fig. 56*b* is not to be recommended for this form of arch.

If the truss of Fig. 56*a* is made fixed in position at points *a* and *b* by pins about which the truss may freely rotate from any cause, the structure becomes a two-hinged arch. Fixing the position of the points *a* and *b*, however, prevents the structure from undergoing the same change in shape under the elastic or temperature changes in length of its members that can occur for either the simple truss or three-hinged arch.

If the truss of Fig. 56*a* is made fixed at points *a*, *b*, *d* and *e* by pins, the structure becomes an arch without hinges. The truss now cannot rotate as a whole about points *a* and *b*, due to fixing the position of points *d* and *e*.

The form of truss here used for illustration would not be satisfactory for an arch without hinges for most locations, owing to the difficulty of fixing the points d and e by a sufficient anchorage. The outline of truss for the arch without hinges, the method of computing the stresses which is given in Art. 62, will not be the one here selected to illustrate the method of computing the stresses given in the five following articles for an arch with two and three hinges.

ART. 57. LOADING AND UNIT STRESSES

In the four articles following this the stresses and cross-sections for all the members for an arch with three and two hinges for the outline of Fig. 56a will be given.

The computation and design for the three-hinged arch will be referred to as Case I, and for the two-hinged arch as Case II.

These arches are designed for a live load of 2160 lbs. per lineal foot, or 36,000 lbs. per panel of the arch, and a dead load of 2880 lbs. per lineal foot, or 48,000 lbs. per panel of the arch. This large dead load is such as would be due to a heavy asphalt floor carried by buckle plates. All the dead load has been taken as applied to the panel points of the top chord.

No metal less than $\frac{5}{16}$ in. in thickness is used for either case; and only two sizes of channels—12 and 15 ins.—are used for the members other than the arch ring. By introducing another shape, say 10-in. channels, a small saving of weight could have been made in both cases, but the appearance of the arch is better on account of the greater uniformity, and the construction is cheapened for the same reason.

The unit stresses used in proportioning the members were:

For tension:

Live-load stresses, 11,000 lbs. per square inch.

Dead-load " 22,000 " "

For compression, in arch ring and top chord:

Live-load stresses, $12{,}000 - 55 \frac{l}{r}$ lbs. per square inch.

Dead-load " $24{,}000 - 110 \frac{l}{r}$ " "

In web members:

Live-load stresses, $11{,}000 - 50 \frac{l}{r}$ lbs. per square inch.

Dead-load " $22{,}000 - 100 \frac{l}{r}$ " "

It would perhaps have been more consistent and better to have used for the arch rings and top chords, the compression formulas specified for web members, as the loading producing maximum stress for the members in these groups is partial, that is, not covering the entire span, for almost all the members of both groups.

These unit stresses, as will be recognized, are those for medium steel, of Theodore Cooper's Specifications of 1896 for Highway Bridges.

The temperature stresses in the two-hinged arch are treated as dead-load stresses, which is thought to be a fair assumption. Members subject to alternate stresses of tension and compression were proportioned to resist both kinds of stress, with eight-tenths of the smaller added to either.

ART. 58. THE THREE-HINGED ARCH

The stresses for Case I are easily computed by any one of several methods. Their computation is here given, because they show how approximate sections may be determined for a two-hinged arch, and for comparison with those for the two-hinged structure.

For this particular case where the outline of the bottom chord is a parabola the positions of the live load giving maximum stress in any member is readily determined from the well known properties of such arches. In general, however, it is well to construct either an influence drawing as was done for arches with solid ribs in Chapter IV, or a table showing the stress in every member of the bridge for a load of unity placed at each panel point in succession. A table of stresses in the various members, due to unit loads at each panel point, will be used in the chapter to show its great value. An examination of this table will show the panel points which are to be loaded to produce maximum stresses of either tension or compression in any member. Such a table is of great use in computing the actual dead and live-load stresses. The method of computing stresses by means of such a table is a very old one, and is to be recommended for any structure of great magnitude or unusual form. In such structures a correct assumption of both the amount and distribution of the dead load is rarely correctly made. With a table of the stresses in every member due to a load of unity at each panel point, all subsequent stress calculations are a simple matter and a revision of the dead-load stresses is not a matter of such magnitude as to prevent it being made.

To properly illustrate the ease and simplicity of the method the table of stresses in the members for the left half of the three-hinged arch of Fig. 58a has been partly made.

The stresses in the members should be determined for a load of unity placed at

point No. 7, as the first operation. These stresses should be computed analytically and to one decimal place beyond that recorded.

The stresses in all the members due to a load of unity at points 1, 2, 3, 4, 5 and 6 in succession are to be taken as a ratio of the respective stresses for a load at point 7.

The stresses in all the members due to loads at points 8, 9, 10, 11, 12 and 13 are to be computed first for the horizontal components of the outer forces, and these stresses are always a simple ratio of the stress due to the horizontal component of the reaction for the center load, No. 7, and second for the vertical component of the reaction and the load. For this case where the member is to the left of the load the stress desired is a simple ratio of the stress in the same member due to the

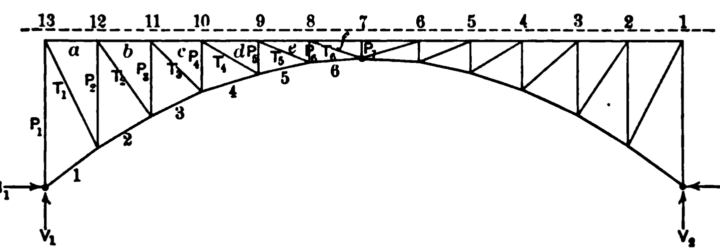


FIG. 58a.

center load, and where the member is to the right of the load it is similar ratio plus or minus the effect of the load.

Table No. 58a has been completed for members c, 3, T_3 and P_3. The computation of such a table of stresses can be greatly facilitated by the use of a Thatcher or other long slide rule. With a table similar to No. 58a for any structure the live load, no matter what its nature, may be readily placed to give the maximum stresses of tension and compression. Such a table may be called an influence table and is easier to make than would be the construction of the influence lines for *stress* for all the members. It is not as easy to construct as the influence line drawing for *position of loading* to produce maximum stresses, but as such a drawing only shows the portion or portions of the structure which are to be loaded to produce maximum stresses of either tension or compression, and is of no great value in their analytical computation, the stresses will be computed by the use of an influence table for unit loads. The influence line drawing for *position of loadings* required to produce maximum stresses in the various members is shown in Fig. 58b.

As a single example of its use let it be required to show which panel points are to be loaded to produce maximum live-load compression in the top chord member c. Pass a section as shown by the dotted line. The center of moments to

Table No. 58*a*

REACTION COMPONENTS AND STRESSES DUE TO PANEL LOADS OF UNITY FOR CASE I

Load at point number.	Horizontal and vertical components of the reactions.			Stresses in members, due to the horizontal and vertical components of the reactions, and their combined effects which are the stresses due to the load.					
		Left	Right	*a*	*b*	*c*	*d*	*e*	*f*
1	*H* *V*	0.000 0.000 0/12				 +0.000			
2	*H* *V*	0.208 0.083 1/12				 +0.130			
3	*H* *V*	0.417 0.167 2/12				 +0.260			
4	*H* *V*	0.625 0.250 3/12				 +0.390			
5	*H* *V*	0.833 0.333 4/12				 +0.521			
6	*H* *V*	1.042 0.417 5/12				 +0.651			
7	*H* *V*	1.250 0.500	1.250 0.500 6/12	+0.452 −0.247 +0.205	+1.168 −0.701 +0.467	+2.344 −1.563 +0.781	+4.255 −3.191 +1.064	+6.836 −5.860 +0.976	0.000 0.000 0.000
8	*H* *V*	1.042 0.583	1.042 0.417 5/12			+1.953 −1.823 +0.130			
9	*H* *V*	0.833 0.667	0.833 0.333 4/12			+ 1.562 −2.083 −0.521			
10	*H* *V*	0.625 0.750	0.625 0.250 3/12			+1.172 −2.344 −1.172			
11	*H* *V*	0.417 0.833	0.417 0.167 2/12			+0.781 −1.562 −0.781			
12	*H* *V*	0.208 0.917	0.208 0.083 1/12			+0.391 −0.781 −0.390			
13	*H* *V*	0.000 1.000	0.000 0.000			+0.000 −0.000			
			$\Sigma+S=$ $\Sigma-S=$ $\Sigma S=$			+2.863 −2.864 −0.001			

TABLE NO. 58a—*Continued*

Load at point number.	Horizontal and vertical components of the reactions.			Stresses in members, due to the horizontal and vertical components of the reactions, and their combined effects, which are the stresses due to the load.					
		Left	Right	1	2	3	4	5	6
1	H V					0.000			
2	H V					−0.316			
3	H V					−0.632			
4	H V					−0.948			
5	H V					−1.264			
6	H V					−1.580			
7	H V	1.250 0.500	1.250 0.500	−1.550 +0.000 −1.550	−1.985 +0.288 −1.697	−2.670 +0.774 −1.896	−3.887 +1.647 −2.240	−5.615 +3.255 −2.360	−8.104 +5.872 −2.232
8	H V	1.042 0.583		1.292	1.653	−2.225 +0.903 −1.322	3.235	4.685	6.750
9	H V	0.833 0.667				−1.780 +1.032 −0.748			
10	H V	0.625 0.750				−1.335 +1.161 −0.174			
11	H V	0.417 0.833				− .890 +1.290 +0.400			
12	H V	0.208 0.917				−0.445 +0.645 +0.200			
13	H V	0.000 1.000				−0.000 +0.000			
			$\Sigma +S=$ $\Sigma -S=$ $\Sigma\ S=$			+0.600 −8.880 −8.280			

TABLE No. 58a—*Continued*

Load at point number.	Horizontal and vertical components of the reactions.			Stresses in members, due to the horizontal and vertical components of the reactions, and their combined effects, which are the stresses due to the load.					
		Left.	Right.	T_1	T_2	T_3	T_4	T_5	T_6
1	H V					−0.000			
2	H V					−0.072			
3	H V					−0.145			
4	H V					−0.218			
5	H V					−0.290			
6	H V					−0.363			
7	H V	1.250 0.500	1.250 0.500	−1.022 +0.558 −0.464	−1.247 +0.791 −0.456	−1.629 +1.194 −0.435	−2.256 +1.922 −0.334	−2.806 +2.900 +0.094	−1.592 +2.630 +1.038
8	H V	1.042 0.583		−0.852	1.038	−1.357 +1.393 +0.036	1.878	2.335	1.325
9	H V	0.833 0.667				−1.086 +1.592 +0.506			
10	H V	0.625 0.750				−0.814 +1.791 +0.977			
11	H V	0.417 0.833				−0.543 +0.545 +0.002			
12	H V	0.208 0.917				−0.272 +0.273 +0.001			
13	H V	0.000 1.000				0.000 0.000			
			$\Sigma+S=$ $\Sigma-S=$ $\Sigma S=$			+1.522 −1.523 −0.001			

TABLE No. 58*a*—*Continued*

Load at point number.	Horizontal and vertical components of the reactions.			Stresses in members, due to the horizontal and vertical components of the reactions, and their combined effects, which are the stresses due to the load.						
		Left.	Right.	P_1	P_2	P_3	P_4	P_5	P_6	P_7
1	H V					+0.000				
2	H V					+0.050				
3	H V					+0.101				
4	H V					+0.151				
5	H V					+0.201				
6	H V					+0.251				
7	H V	1.250 0.500	1.250 0.500	+0.917 −0.500 +0.417	+1.021 −0.648 +0.373	+1.129 −0.827 +0.302	+1.198 −1.021 +0.177	+1.101 −1.138 −0.037	+0.539 −0.891 −0.352	+0.000 −1.000 −1.000
8	H V	1.042 0.583				+0.940 −0.965 −0.025				
9	H V	0.833 0.667				+0.752 −1.104 −0.351				
10	H V	0.625 0.750				+0.564 −1.240 −0.675				
11	H V	0.417 0.833				+0.376 −1.378 −1.002				
12	H V	0.208 0.917				+0.188 −0.189 −0.001				
13	H V	0.000 1.000				+0.000 −0.000				
			$\Sigma +S=$ $\Sigma -S=$ $\Sigma S=$			+1.056 −2.056 −1.000				

be selected for stress in the member *c* is the point *k*. Inspection of Fig. 58*b* shows that the loads tending to produce clockwise rotation of the part of the arch to the left of the section formed by the dotted line and therefore compression in the member *c*, are P_9 to P_{12} inclusive. By means of this drawing (Fig. 58*b*) the loadings which produce maximum tensile or compressive stress in each member of the arch may be determined and tabulated much the same as was done in Arts. 37 and 39.

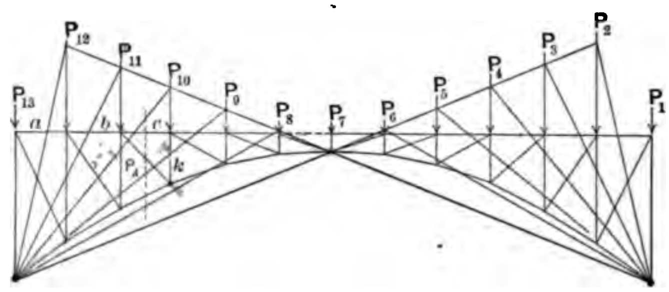


FIG. 58*b*.

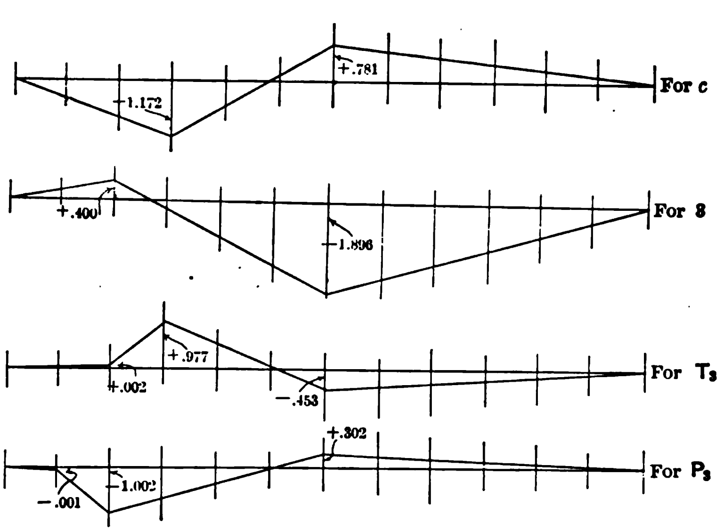


FIG. 58*c*.

Influence lines for stress show the effect of a load at any point in producing stress in a given member, and while they are most valuable to the student in showing the varying stress producing effect for a given member of a load as it passes over the structure, their construction does not facilitate the stress computation for a simple structure such as the one under consideration. As all the data are at hand for constructing influence lines for stress in all the members, the influence lines for stress for four members by way of illustration are shown in Fig. 58*c*.

For any structure which is to carry a system of moving concentrated loads the influence table shows just as clearly as do influence lines, the part of the span to be loaded and where to place the heaviest loads for maximum stresses. The panel loads, resulting from the correct placing of the rolling load, multiplied by the stresses due to the unit loads of the table, give the maximum stresses of the various members.

Table No. 58*b* gives the live- and dead-load stresses, cross-sections of member and weight for the Arch of Case I.

TABLE NO. 58*b*

Member	Live-load Stresses	Dead-load Stresses	Sections	Weight in Pounds
			Sq. Ins.	
P_1	+ 59000 − 77000	− 24000	2 15″ channels 33.3 = 19.6	
P_2	+ 50400 − 86400	− 48000	2 12″ " 29.7 = 17.4	
P_3	+ 37900 − 73900	− 48000	2 12″ " 21.7 = 12.7	
P_4	+ 24700 − 60700	− 48000	2 12″ " 21.7 = 12.7	14230
P_5	+ 11700 − 47700	− 48000	2 12″ " 21.7 = 12.7	
P_6	+ 30600 − 66600	− 48000	2 12″ " 21.7 = 12.7	
P_7	− 36000	− 48000	2 12″ " 21.7 = 12.7	
T_1	± 65800	....	2 12″ " 33.3 = 19.6	
T_2	± 61600	....	2 12″ " 26.7 = 15.7	
T_3	± 54900	....	2 12″ " 21.7 = 12.7	
T_4	± 51100	....	2 12″ " 21.7 = 12.7	15560
T_5	± 49600	....	2 12″ " 21.7 = 12.7	
T_6	±130800	....	2 12″ " 46 = 27.1	
a	± 29200	....	2 15″ " 32 = 18.8	
b	± 64500	....	2 15″ " 32 = 18.8	
c	±103100	....	2 15″ " 32 = 18.8	
d	±134300	....	2 15″ " 41.3 = 24.3	14230
e	±122800	....	2 15″ " 41.3 = 24.3	
f	....	....	2 15″ " 32 = 18.8	
1	−334800	−446400	4 Ls. 4×4×16.0 2 pls. 24×$\frac{11}{16}$ = 51.8	
2	+ 7100 −321900	−419800	4 Ls. 4×4×14.7 2 pls. 24×$\frac{11}{16}$ = 50.3	
3	+ 21600 −319600	−397300	4 Ls. 4×4×16.0 2 pls. 24×$\frac{5}{8}$ = 48.8	
4	+ 41500 −326200	−379500	4 Ls. 4×4×14.7 2 pls. 24×$\frac{5}{8}$ = 47.3	36180
5	+ 53700 −329200	−367100	4Ls. 4×4×14.7 2 pls. 24×$\frac{5}{8}$ = 47.3	
6	+ 10600 −281200	−360800	4 Ls. 4×4×14.7 2 pls. 24×$\frac{9}{16}$ = 44.3	
			Details (from shop drawings).......	34800
			Total weight =	115000

PROBLEMS

No. **58*a***. From the drawing of Fig. 58*b* select the points to be loaded for maximum stress in member P_3.

No. **58*b***. Complete the Table No. 58*a*.

ART. 59. THE TWO-HINGED ARCH, THE FIRST METHOD OF COMPUTATION

The center hinge c of Case I will be now considered removed. This would practically be accomplished by connecting the member f at both ends, thereby enabling the structure to develop a moment at this point and preventing the free motion of the points 6 and 8 of Fig. 59a either from approaching each other as indicated in Fig. 59b or from moving away from each other as indicated in Fig. 59c. Let the dotted lines of both Figs. 59b and 59c show the normal figure. Now if the members f and f_0 of Fig. 59a are considered removed and the temperature falls, or a live load comes on the arch, the points 6 and 8 come nearer together as is shown in Fig. 59b; if the temperature rises the points 6 and 8 move away from each other as is shown in Fig. 59c. It is at once seen that introducing elastic members f and f_0 tends to restrain this motion and that the stress in f_0 and f will be such as is produced by connecting it to points 6 and 8.

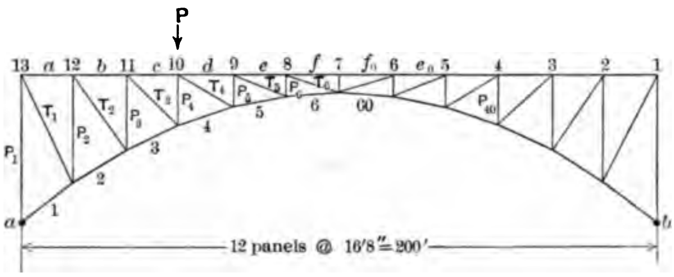


FIG. 59a.

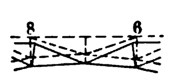


FIG. 59b.

FIG. 59c.

Case II.—To find the stresses in the members of the arch of Fig. 59a, constructed with hinges at a and b, due to a load P at any point, consider the members f and f_0 cut at point 7 and compute the stresses as for a three-hinged arch. The manner of doing this has been fully illustrated in the previous article. Then compute the horizontal motion Δ of the two cut ends with reference to each other under the load P.

Then compute the horizontal motion δ of the two cut ends with reference to each other for a horizontal force of unity at the cut ends.

The stress S_f in f for the two cut ends brought together is $S_f = \frac{\Delta}{\delta}$.

If S_2 = stress in any member of the truss when considered as a two-hinged arch; and

S_3 = stress in the member when the structure is considered to have three hinges;

and

S_c = stress in the member, when the structure is considered to have three hinges, due to S_f.

Then

$$S_2 = S_3 \pm S_c.$$

To find the temperature stresses

let Δ_t = horizontal motion under the given change of temperature of the cut ends of f with reference to each other, and

S_{ft} = temperature stress in f.

Then

$$S_{ft} = \frac{\Delta_t}{\delta},$$

in which δ is as before defined.

Having the stress in the member f for a load at any point of the arch or for a temperature change of any amount, the complete stress computation and design is a simple matter. With a design completed for the structure under the assumption that three hinges will be used, that is for f and f_0 cut, the method just indicated leads to a very easy solution for the ends of f and f_0 connected or the arch with two hinges.

Table 59*a* gives a good form for arranging the computations for the necessary deflections. The notation and arrangement of the table need no explanation beyond that of the foot note. It will be seen that the column headed $\frac{TL}{EA}$ gives the change in length of each member due to a horizontal force acting toward each of the cut ends at point No. 7, or due to a compression of unity in f and f_0.

Having the stresses S_7 to S_{13} the partial deflections for each member of the left half of the arch may be obtained from one setting of the slide rule, while the partial deflection due to the entire right half of the arch will be a simple proportion of that due to the load at the center.

This manner of solving this problem by selecting as the indeterminate stress that in the member f makes the analytical method of determining the necessary deflections not only the most accurate, as it always is, but the quickest, there being no member of the arch which is not subject to both change of position and rotation

TABLE

Marks.	Length, Ft.	Area, Sq.in.	$\frac{L}{A}$	T	$\frac{TL}{EA}$	$\frac{T^2L}{EA}$	S_7	$\frac{S_7TL}{EA}$	S_8	$\frac{S_8TL}{EA}$
a	16.6667	18.8	0.89	−0.054	−0.017	+0.001	+0.205	−0.003		
b	16.6667	18.8	0.89	−0.136	−0.042	+0.006	+0.467	−0.020		
c	16.6667	18.8	0.89	−0.282	−0.087	+0.024	+0.781	−0.068		
d	16.6667	24.3	0.69	−0.511	−0.122	+0.062	+1.064	−0.130		
e	16.6667	24.3	0.69	−0.820	−0.195	+0.160	+0.976	−0.191		
f	16.6667	18.8	0.89	−1.000	−0.307	+0.307	+0.000	−0.000		
1	20.6679	51.8	0.40	+0.181	+0.025	+0.005	−1.550	−0.039		
2	19.4365	50.3	0.39	+0.238	+0.032	+0.008	−1.697	−0.054		
3	18.3922	48.8	0.38	+0.320	+0.042	+0.014	−1.896	−0.080		
4	17.5682	47.3	0.37	+0.466	+0.060	+0.028	−2.240	−0.134		
5	16.9967	47.3	0.36	+0.673	+0.084	+0.057	−2.360	−0.198		
6	16.7037	44.3	0.38	+0.973	+0.127	+0.124	−2.232	−0.283		
T_1	37.6546	19.6	1.92	+0.123	+0.081	+0.010	−0.464	−0.038		
T_2	29.0355	15.7	1.85	+0.150	+0.096	+0.013	−0.456	−0.044		
T_3	23.1036	12.7	1.82	+0.193	+0.121	+0.023	−0.435	−0.053		
T_4	19.6689	12.7	1.55	+0.271	+0.145	+0.039	−0.334	−0.048		
T_5	18.3942	12.7	1.45	+0.336	+0.168	+0.057	+0.094	+0.016		
T_6	17.7138	27.1	0.65	+0.191	+0.045	+0.008	+1.038	+0.045		
P_1	46.0000	19.6	2.34	−0.110	−0.089	+0.010	−0.417	−0.037		
P_2	33.7778	17.4	1.94	−0.122	−0.082	+0.010	+0.373	−0.031		
P_3	23.7778	12.7	1.88	−0.136	−0.088	+0.012	+0.302	−0.026		
P_4	16.0000	12.7	1.26	−0.144	−0.062	+0.009	+0.177	−0.011		
P_5	10.4444	12.7	0.83	−0.132	−0.038	+0.005	−0.037	+0.001		
P_6	7.1111	12.7	0.56	−0.065	−0.013	+0.001	−0.352	+0.005		
P_7	6.0000	12.7	0.47	−0.000	−0.000	+0.000	−1.000			
						+0.993		−1.421		
a_0					−0.017		+0.205			
b_0					−0.042		+0.467			
c_0					−0.087		+0.781			
d_0					−0.122		+1.064			
e_0					−0.195		+0.967			
f_0					−0.307		+0.000			
10					+0.025		−1.550			
20					+0.032		−1.697			
30					+0.042		−1.896			
40					+0.060		−2.240			
50					+0.084		−2.360			
60					+0.127		−2.232			
T_{10}					+0.081		−0.464			
T_{20}					+0.096		−0.456			
T_{30}					+0.121		−0.435			
T_{40}					+0.145		−0.334			
T_{50}					+0.168		+0.094			
T_{60}					+0.043		+1.038			
P_{10}					−0.089		+0.417			
P_{20}					−0.082		+0.373			
P_{30}					−0.088		+0.302			
P_{40}					−0.062		+0.177			
P_{50}					−0.038		−0.037			
P_{60}					−0.013		−0.352			
						+0.993		−1.421		−1.184
Σ						+1.986		−2.842		

T = the stress in the several members due to unity in f and f_0. S_7, S_8, S_9, S_{10}, S_{11}, S_{12}, and S_{13} are the at 2.9 to avoid long decimals. T_c = change in length due to temperature variation.

No. 59*a*

S_9	$\frac{S_9TL}{EA}$	S_{10}	$\frac{S_{10}TL}{EA}$	S_{11}	$\frac{S_{11}TL}{EA}$	S_{12}	$\frac{S_{12}TL}{EA}$	S_{13}	$\frac{S_{13}TL}{EA}$	T_c	T_cT
								−1.000	+0.089		
									+0.089		
	−0.947		−0.711		−0.474		−0.237		−0.000		
									+0.089		

stresses due to vertical load of unity at points number 7, 8, 9, 10, 11, 12, and 13 respectively. *E* is taken

under the action of a single load, therefore all possible displacement diagrams would need be corrected for rotation. The preceding clearly indicates all the essential steps necessary to compute the final stresses.

PROBLEM

No. 59a. Complete Table No. 59a and by means of it determine the stresses in the member f for a load of one pound at each panel point in succession of the top chord.

ART. 60. THE TWO-HINGED ARCH, SECOND METHOD OF COMPUTATION

The complete design for Case II will now be made by a method which is believed to be a good one for the design of an entirely new example, where the dimensions of the structure are not so great as to make graphical methods of determining the necessary deflections unreliable. For structures of large dimensions all deflections should be computed analytically. The difficulty of making a displacement diagram together with a rotation diagram on any drawing of possible size for a structure, say 500 ft. long, is too great an undertaking for this method to give good results. The graphical method of finding deflections should be thoroughly understood, however, and as it is very good and accurate for a structure of this size it is here used.

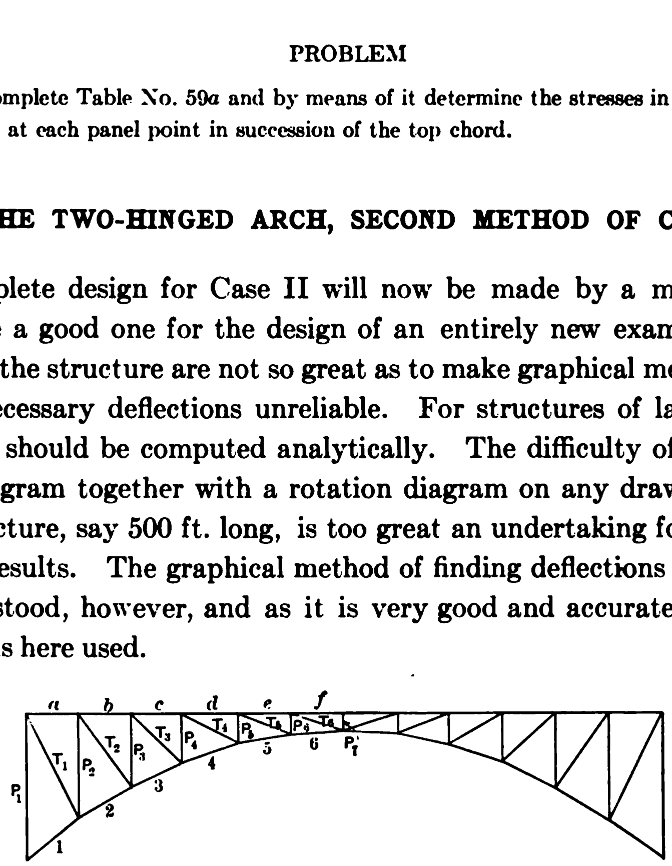


FIG. 60a.

If in Fig. 60a the right end be designated b and be supposed to rest on a support which allows free horizontal motion, then under a horizontal force the span will be deformed in such a way that the center vertical still remains vertical. A displacement diagram for any horizontal force at b may be made, by starting out from point 7 and member P_7, which need not be corrected for rotation, and as the distorted truss will be symmetrical about this member only one-half of the diagram need be constructed. It can be readily seen by the aid of Maxwell's Theorem that: a vertical load of unity at any point x produces a horizontal motion at b, which is equal to the vertical motion of x produced by a horizontal force of unity at b.

The vertical components of the reactions for any load are the same as the

reactions for a simple truss of the same span. The horizontal component of the reactions for any load can be found from the following relation:

$$H = \frac{P\delta}{\delta_1},$$

in which H = Horizontal thrust produced by P;

P = Load at any point;

δ = Vertical deflection of the loaded point, due to a horizontal force of unity acting at the free hinge—one hinge assumed fixed and the other free;

δ_1 = Horizontal displacement of the free hinge, due to a horizontal force of unity acting at the free hinge.

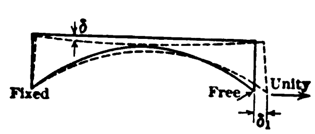


FIG. 60b.

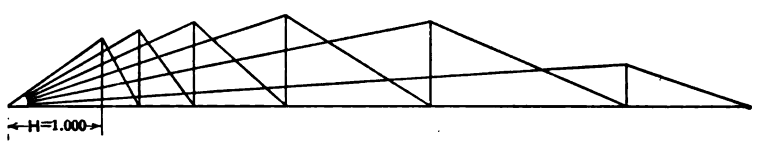


FIG. 60c.

Fig. 60b will make the notation of this formula more clear than the definitions alone could.

For simplicity of comparison the results are arranged as compactly as possible in Table No. 60a.

The first column gives the marks designating the members (see Fig. 60a). The second column shows the stresses due to a horizontal force of unity acting at one of the hinges. These stresses should be very carefully computed analytically, and then checked graphically by means of such a diagram as Fig. 60c. From these stresses the changes in length e of the various members were computed from the formula $e = \frac{TL}{EA}$. In determining these values of e, which are given in the third column, L was taken in feet, A, at present unknown, was taken equal to unity, and E was taken at 29 instead of 29,000,000, in order to give values of e sufficiently large to allow them to be accurately plotted. These preliminary values of e are 1,000,000 $\times A$ times their true value. Using these values of e the displacement diagram shown in dotted lines in Fig. 60d was constructed. The full lines in this figure give the final displacement diagram, and from this diagram the preliminary ratio of δ to δ_1 was found to have the following values, beginning with the panel point at the end of the arch: 0.003, 0.203, 0.401, 0.590, 0.764, 0.904 and 0.962.

The thrust due to a load at the end of the span being small, only 0.003 of the load, it is neglected in both the preliminary and final calculations.

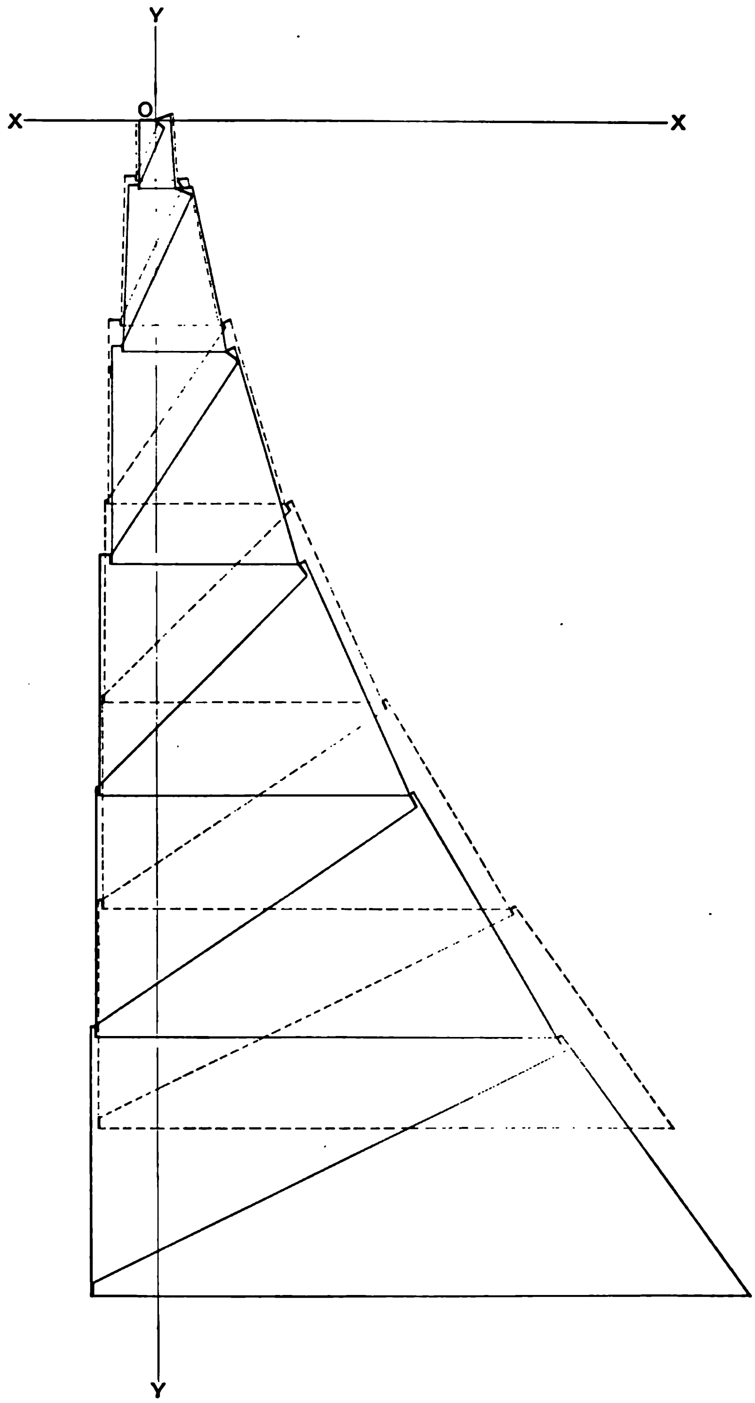


FIG. 60d.

For full loading the preliminary value of the horizontal thrust is $6.686 \times 48{,}000$ lbs.; from this, by means of a diagram similar to Fig. 60*e* (Fig. 60*e* being the final dead-load stress diagram), the preliminary dead-load stresses are obtained. These stresses are given in the sixth column of the table.

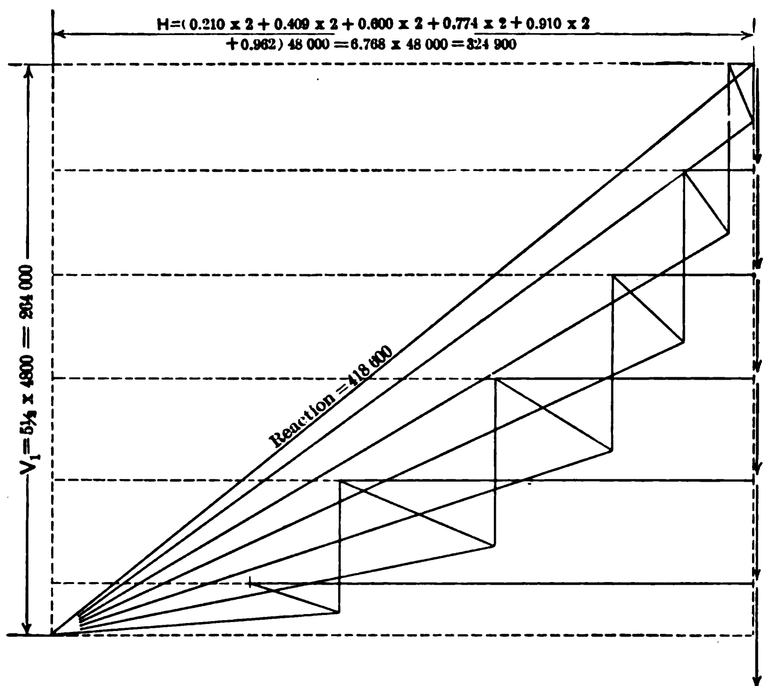


FIG. 60*e*.

Before finding the preliminary live-load stresses it will be well to find the stresses in the arch due to a vertical reaction of unity, supposed to be applied in this case at the left hinge. The stresses due to this load are given in the fifth column

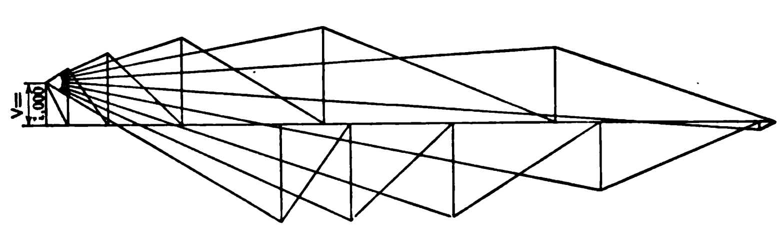


FIG. 60*f*.

of the table. They were computed analytically and checked by means of the diagram, Fig. 60*f*. The upper figures in the fifth column, where two sets are given, are the stresses for the corresponding member in the right half of the arch.

TABLE

CASE II. STRESSES, SECTIONS, WEIGHT, ETC., FOR A TWO-HINGED ARCH, HAVING

Truss Members	Stresses due to H.	e Preliminary	e Final.	Stresses due to V.	Dead-load Stresses. Preliminary.	Dead-load Stresses. Final.	Live-load Stresses. Preliminary.	Live-load Stresses. Final.
P_1	+0.733	+1.162	+0.062	− 1.00	− 53000	− 49800	+ 40700 − 80300	+ 42000 − 79400
P_2	+0.817	+0.952	+0.051	− 2.257 − 1.296	− 80000	− 76900	+ 30800 − 90800	+ 32000 − 89600
P_3	+0.903	+0.740	+0.044	− 2.271 − 1.654	− 85200	− 79300	+ 20100 − 82500	+ 21400 − 81100
P_4	+0.958	+0.529	+0.039	− 2.125 − 2.042	− 86000	− 82000	+ 8500 − 72700	+ 10500 − 71900
P_5	+0.881	+0.317	+0.025	− 1.553 − 2.277	− 82700	− 79000	+ 9900 − 71600	+ 10700 − 69900
P_6	+0.431	+0.106	+0.008	− 0.094 − 1.781	− 64600	− 62800	+ 27600 − 76200	+ 28100 − 75500
P_7	....	....	....	....	− 48000	− 48000	− 36000	− 36000
T_1	−0.818	−1.062	−0.084	+ 1.115	+ 32200	+ 28800	− 45500 + 69500	− 46900 + 68400
T_2	−0.998	−0.999	−0.079	+ 2.756 + 1.583	+ 38900	+35300	− 37000 + 66000	− 39100 + 65400
T_3	−1.304	−1.039	−0.082	+ 3.279 + 2.389	+ 53000	+ 45500	− 29000 + 67200	− 30900 + 65100
T_4	−1.805	−1.224	−0.097	+ 4.002 + 3.845	+ 70900	+ 63600	− 19000 + 72000	− 22800 + 70300
T_5	−2.245	−1.403	−0.095	+ 3.958 + 5.801	+ 88000	+ 78300	− 39400 +105600	− 42400 +101400
T_6	−1.273	−0.778	−0.033	+ 0.277 + 5.259	+ 49300	+ 43000	−118700 +155700	−122800 +154200
a	+0.362	+0.208	+0.011	− 0.493	− 14500	− 12700	+ 20100 − 30700	+ 20800 − 30200
b	+0.935	+0.537	+0.029	− 7.009 − 1.402	− 36500	− 33200	+ 41400 − 68700	+ 43000 − 67500
c	+1.875	+1.078	+0.057	− 9.375 − 3.125	− 74500	− 66300	+ 61000 −115500	+ 64000 −113400
d	+3.404	+1.956	+0.059	−12.766 − 6.383	−134100	−120000	+ 66500 −166300	+ 71500 −160900
e	+5.469	+3.143	+0.095	−16.406 −11.719	−214700	−192000	+ 36100 −196300	+ 44200 −188300
f	+6.667	+3.831	+0.107	−16.667	−260900	−232800	−195400	+ 800 −176400
1	−1.240	−0.884	−0.018	+ 0.000	−398000	−403000	−298500	−302100
2	−1.588	−1.064	−0.023	+ 0.575	−357600	−364000	+ 7400 −275700	+ 7000 −281200
3	−2.135	−1.354	−0.033	+ 7.735 + 1.547	−313200	−321800	+ 23200 −258600	+ 22000 −260200
4	−3.031	−1.836	−0.047	+ 9.882 + 3.294	−259900	−272900	+ 47600 −243500	+ 44900 −249700
5	−4.492	−2.633	−0.083	+13.019 + 6.509	−190600	−209100	+ 74000 −217000	+ 68300 −224700
6	−6.483	−3.735	−0.117	+16.442 +11.745	−107000	−133400	+ 72000 −152500	+ 62400 −162100

* The temperature stresses in the final calculations were found

No. 60*a*

A SPAN LENGTH OF 200 FEET FROM CENTER TO CENTER OF END PINS

Temperature Stresses. Preliminary and Final.*	Section. Preliminary.	Section. Final.	Weight of Final Sections in Pounds.
±7300	2 15-in. channels 96 = 18.8 sq.ins.	2 15-in. channels 96 = 18.8 sq.ins.	
8200	2 15-in. " 96 = 18.8 "	2 15-in. " 96 = 18.8 "	
9000	2 12-in. " 85 = 16.7 "	2 12-in. " 82 = 16.1 "	
9600	2 12-in. " 70 = 13.7 "	2 12-in. " 69 = 13.5 "	
8800	2 12-in. " 65 = 12.7 "	2 12-in. " 65 = 12.7 "	
4300	2 12-in. " 65 = 12.7 "	2 12-in. " 65 = 12.7 "	
....	2 12-in. " 65 = 12.7 "	2 12-in. " 65 = 12.7 "	15220
8200	2 12-in. " 65 = 12.7 "	2 12-in. " 65 = 12.7 "	
10000	2 12-in. " 65 = 12.7 "	2 12-in. " 65 = 12.7 "	
13000	2 12-in. " 65 = 12.7 "	2 12-in. " 65 = 12.7 "	
18000	2 12-in. " 65 = 12.7 "	2 12-in. " 65 = 12.7 "	
22500	2 12-in. " 79 = 15.5 "	2 12-in. " 75 = 14.7 "	
12700	2 12-in. " 120 = 23.5 "	2 12-in. " 121 = 23.7 "	13430
3600	2 15-in. " 96 = 18.8 "	2 15-in. " 96 = 18.8 "	
9400	2 15-in. " 96 = 18.8 "	2 15-in. " 96 = 18.8 "	
18800	2 15-in. " 96 = 18.8 "	2 15-in. " 96 = 18.8 "	
34000	2 15-in. " 169 = 33.2 "	2 15-in. " 159 = 31.2 "	
55000	2 15-in. " 169 = 33.2 "	2 15-in. " 159 = 31.2 "	
66700	2 15-in. " 183 = 35.9 "	2 15-in. " 166 = 32.6 "	17890
12400	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times51 \\ \text{2 pls. } 20\times\frac{3}{4}\end{array}\right\} = 50.0$	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times53 \\ \text{2 pls. } 20\times\frac{3}{4}\end{array}\right\} = 50.8$	
15900	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times51 \\ \text{2 pls. } 20\times\frac{5}{8}\end{array}\right\} = 45.0$	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times53 \\ \text{2 pls. } 20\times\frac{5}{8}\end{array}\right\} = 45.8$	
21400	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times41 \\ \text{2 pls. } 20\times\frac{5}{8}\end{array}\right\} = 41.1$	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times43 \\ \text{2 pls. } 20\times\frac{5}{8}\end{array}\right\} = 41.9$	
30300	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times41 \\ \text{2 pls. } 20\times\frac{9}{16}\end{array}\right\} = 38.6$	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times43 \\ \text{2 pls. } 20\times\frac{9}{16}\end{array}\right\} = 39.4$	
44900	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times29 \\ \text{2 pls. } 20\times\frac{1}{2}\end{array}\right\} = 31.4$	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times29 \\ \text{2 pls. } 20\times\frac{9}{16}\end{array}\right\} = 33.9$	
64800	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times29 \\ \text{2 pls. } 20\times\frac{1}{2}\end{array}\right\} = 31.4$	$\left\{\begin{array}{l}\text{4 Ls. } 4\times4\times29 \\ \text{2 pls. } 20\times\frac{1}{2}\end{array}\right\} = 31.4$	30560
		Details (estimated)	31900
		Total weight	109000

to be the same as in the preliminary calculations.

To determine the greatest tensile and compressive live-load stresses to which any member of the arch is subjected, take the member marked 5 of the arch ring for example, and arrange the computations as follows:

$$
\begin{array}{r|l r|l}
-H & & +V & \\
4.492\times 0.203 & & 13.019\times\frac{1}{12} & \\
\times 0.401 & 1.958 & \times\frac{2}{12} & \frac{5}{6}=10.849; \\
\times 0.590 & & \times\frac{3}{12} & \\
\times 0.764 & 3.432 & \times\frac{4}{12} & 4.340; \\
\times 0.904 & 4.061 & 6.509\times\frac{7}{12} & 3.797; \\
\times 0.962 & & \times\frac{6}{12} & \\
\times 0.904 & & \times\frac{5}{12} & \\
\times 0.764 & 4.728 & \times\frac{4}{12} & 2\frac{1}{3}=15.208. \\
\times 0.590 & & \times\frac{3}{12} & \\
\times 0.401 & & \times\frac{2}{12} & \\
\times 0.203 & & \times\frac{1}{12} &
\end{array}
$$

The value of H and V being those given in the second and fifth columns of Table No. 60*a* opposite 5, by inspection it can be seen that loads on the ninth, tenth, eleventh and twelfth panel points produce tension, and loads on the other points produce compression in the member 5. The computation of the stresses is now very simple, and we have:

For tension $(10.849-4.492\times 1.958)36{,}000=2.054\times 36{,}000=\ \ 74{,}000.$
For compression $(\ 4.728\times 4.492-15.208)36{,}000=6.030\times 36{,}000=217{,}000.$

The other live-load stresses were determined in this manner, and the computations were equally simple. Instead of determining the position of live loading to produce maximum stresses of either tension or compression in the manner just

indicated, the influence loading drawing of Fig. 60*g* may be prepared and used. This drawing is readily made, as the horizontal and vertical component of each reaction is known. The manner of drawing R_1 and R_2 for load P_{12} is indicated on the figure. The lines of action of the reactions for loads at other points are determined in exactly the same manner. Inspection of Fig. 60*g* shows at a glance that loads at the ninth, tenth, eleventh and twelfth panel points produce tension in member 5 and loads at the remaining points compression in this member. The student should note and compare the reaction loci of the influence loading drawings of Figs. 37*c*, 39*e*, 58*b* and 60*g*. The reaction locus is the line formed by joining the points where the reactions for each load intersects the line of action of the load.

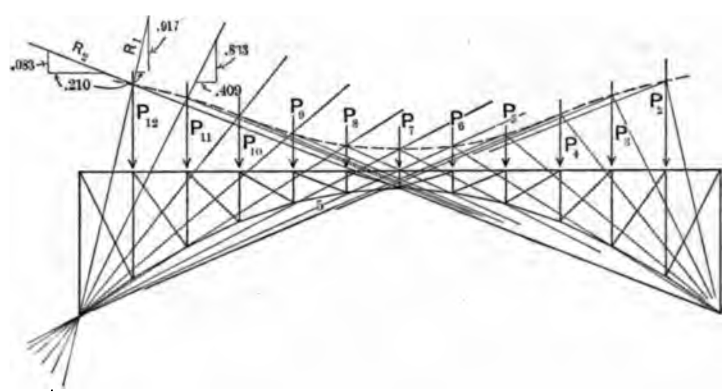


FIG. 60*g*.

Having determined the preliminary live- and dead-load stresses, it now remains to determine the preliminary temperature stresses. From the displacement diagram, already constructed, the horizontal displacement of the hinge due to a horizontal force of unity, is found to be $\frac{230.8}{1{,}000{,}000 \times a}$ ft. It will be assumed that the arch will be subjected to a range of temperature of 120° F., or a variation of 60° from the mean. This is a less variation than is usually assumed, but as the metal of the arch under consideration is to be protected from the direct rays of the sun by a highway floor, it is probably enough. The change in length of the arch due to a variation of 60° F., taking the coefficient of expansion as 0.0000065 per degree F., is 0.078 ft. Dividing this change of length due to temperature by the change in length due to a thrust of 1 lb., we get $\frac{78 \times a \times 10{,}000{,}000 \text{ lbs.}}{1000 \times 2308}$ as the amount of the temperature thrust in which the value of a is at present unknown; the average influence of the areas of all the members of the arch is very nearly represented by the influence of the area of a member of the top chord near the center of the arch. In this case the approximate value of the area of the member e, as deter-

mined from the preliminary live- and dead-load stresses, is used as the value of a in the expression for the temperature thrust. This approximate determination of the area of e gives about 31 sq.ins., which gives for the preliminary temperature thrust $\frac{78\times 31\times 10{,}000{,}000}{1000\times 2308}=10{,}000$ lbs. The preliminary temperature stresses are now determined by multiplying the stresses in the second column of the table by 10,000.

From these preliminary dead-load, live-load and temperature stresses the preliminary sections were determined.

Using these new areas, instead of unity used in the preliminary calculation, the final calculations of stresses are made in exactly the same manner as the preliminary calculations. It is found that the preliminary temperature stresses need not be changed for the final determination of the sections. A final determination of the sections shows only small changes from the preliminary. In most of the members there is no change, and the greatest change amounts to only 6 per cent, and as the preliminary determination for this case was on the safe side it seems that the final calculation was hardly necessary. Where an estimate only is required, the preliminary calculation would certainly be ample.

PROBLEMS

60a. By means of Fig. 60g find the points to be loaded to produce maximum compression in bottom chord 1.

60b. By means of Table No. 60a find the points to be loaded to produce maximum comcompression in bottom chord 1.

ART. 61. COMPARISON OF THE WEIGHT OF METAL REQUIRED TO CONSTRUCT THE ARCH WITH TWO AND THREE HINGES

The calculations of weight give:

For Case I. . . . . 115,000 lbs.
For Case II. . . . . 109,000 "

This gives a saving in weight of 5½ per cent in favor of the two-hinged arch. A variation of 75° in temperature would have lessened the difference considerably, but the two-hinged arch would still have been lighter.

While it is hardly allowable to draw general conclusions from the consideration of a special case, it can be said that where an arch of this form (spandrel-braced) is suitable, the two-hinged arch is lighter than the three-hinged arch; it is also cheaper to construct, as there is no center pin and there are no adjustable

members at the center of the arch. The floor system of the two-hinged arch would also be more simple than that of the three-hinged arch, for the great range of height of the center-pin of the three-hinged arch, due to temperature and live-load stresses, makes a troublesome break in the floor system at this point.

When spandrel-braced arches are used in series, supported on intermediate masonry piers, the two-hinged arch has the advantage of having less horizontal thrust, and therefore requiring smaller piers than the three-hinged arch with the center pin in the bottom chord . Great care must be taken in the construction of the masonry for the two-hinged arch, in order that it may not be subject to even slight settlement or displacement; but, taking this extra work into account, it is believed that the masonry for a series of two-hinged arches will cost less than the masonry for a series of three-hinged arches.

ART. 62. THE OPEN-WEBBED ARCH WITHOUT HINGES. LIVE- AND DEAD-LOAD STRESSES

For the purpose of illustration, assume an arch rib as shown in Fig. 62*a*. This structure when loaded has four unknown reactions or outer forces applied to it in addition to the given load. From the nature of the supports indicated it

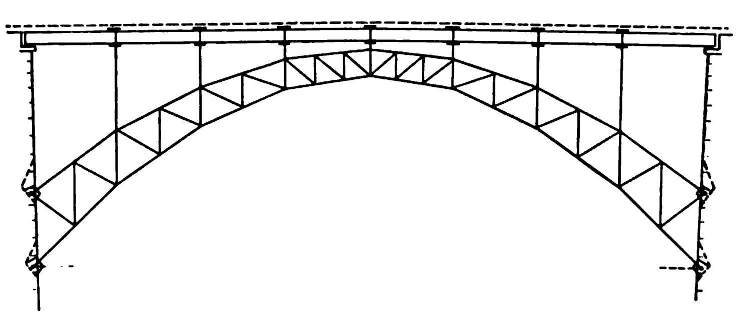


FIG. 62*a*.

can be seen that both the direction and magnitude of the upper reaction at each end are unknown, and that only the magnitude of the lower reaction at each end is unknown. Expressions for the unknowns of the reactions in terms of certain deflections due to unit loads can readily be written. It is seen, however, that the computation of the necessary unit load stresses and deflections would be very laborious. A method of computing the stresses in such an arch when it is symmetrical about its vertical center line will now be developed in detail. Consider the rib cut at the center by a vertical plane; this divides the rib into two parts, as shown in Figs. 62*b* and 62*c*, the vertical member at the center being con-

sidered divided into equal parts, one of which is taken with Fig. 62*b* and the other with Fig. 62*c*.

It is perfectly clear that a load at any point on the left half produces stresses in the truss members of that half which may be determined by statics when each half acts as an independent cantilever truss, and that no stresses can be produced in the right half by this load when the two halves are not connected.

If we suppose the two halves to be connected by joining the points, *a* and *b*, then, under any condition of loading, these two points have the same motion, as they are the same point. Therefore, for a load on the left half it is clear that the stress in any member of that half is that due to the given load when the half is considered as a separate structure modified by the stress acting between *a* and *b* caused by connecting these points, and the stress in any member of the right half is that due to the stress acting between *a* and *b* caused by connecting these points.

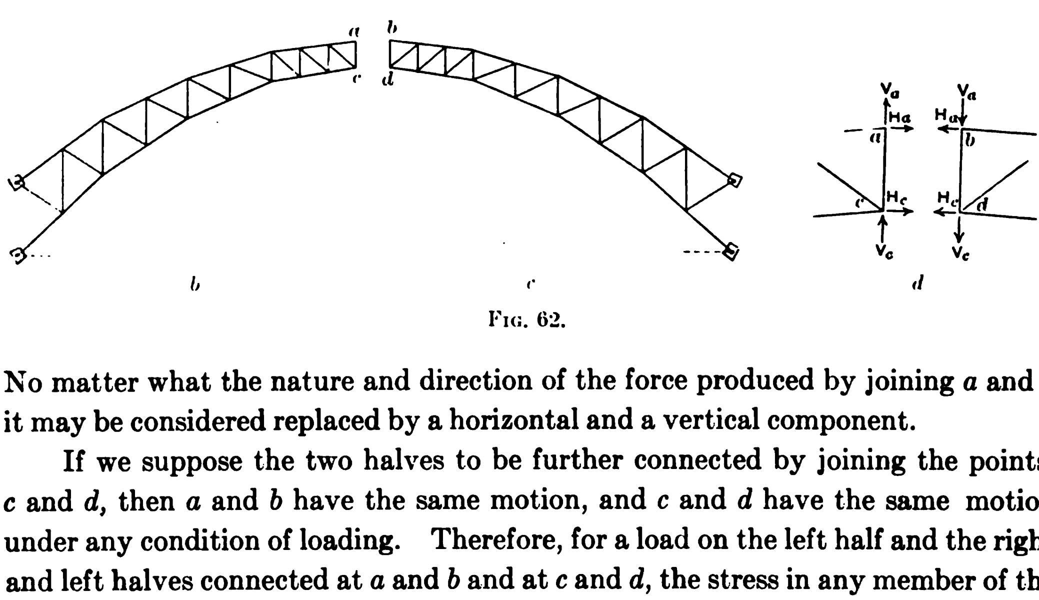


FIG. 62.

No matter what the nature and direction of the force produced by joining *a* and *b* it may be considered replaced by a horizontal and a vertical component.

If we suppose the two halves to be further connected by joining the points, *c* and *d*, then *a* and *b* have the same motion, and *c* and *d* have the same motion under any condition of loading. Therefore, for a load on the left half and the right and left halves connected at *a* and *b* and at *c* and *d*, the stress in any member of the left half is that due to the load on that half, when the half acts as a simple cantilever modified by the forces produced at *a* and *c* by connecting *a* to *b* and *c* to *d*, and the stress produced in any member of the right half is that due to the forces at *b* and *d* caused by connecting *a* to *b* and *c* to *d*. It hardly need be mentioned that the forces, due to connecting *a* to *b* and *c* to *d*, at *a* and *c*, are equal and opposite to those at *b* and *d*, respectively.

No matter what the amount and direction of the forces acting between *a* and *b*, and *c* and *d*, they may each be replaced by their horizontal and vertical components. For the two halves connected and subjected to any loading, the forces produced by thus joining them may be represented by the four forces of Fig. 62*d*, H_a, V_a, H_c,

and V_c, the direction of which may be as shown or the opposite and the magnitude of which is to be determined. Knowing that for elastic structures with a high modulus of elasticity and under low unit stresses deflections are proportional to the loads that produce them, four equations between the unknown forces, H_a, V_a, H_c, and V_c, and certain easily determined deflections for the half arch acting independently, may readily be written. The solution of the four simultaneous equations will determine the unknown forces acting at the crown of the arch for each half. These equations of condition for finding the unknown forces at the crown will now be written for a symmetrical arch. The general method is equally applicable to an unsymmetrical arch, however.

Let a load of 1 lb., at any point, x, cause the point, a, to take a new position, a', and the point, c, a new position at c', as shown in Fig. 62e.

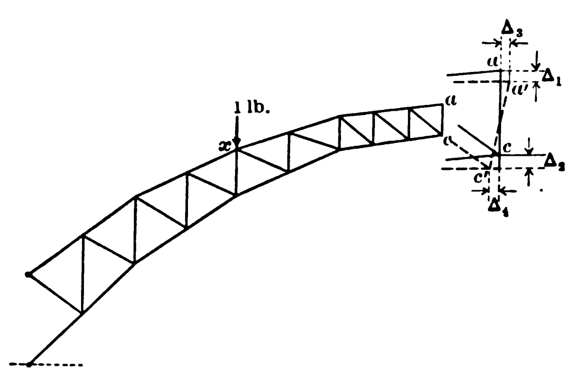


FIG. 62e.

Then

Δ_1 = vertical deflection of a due to a vertical load of unity at any point, x;

Δ_2 = vertical deflection of c due to a vertical load of unity at any point, x;

Δ_3 = horizontal deflection of a due to a vertical load of unity at any point, x;

Δ_4 = horizontal deflection of c due to a vertical load of unity at any point, x;

Let a vertical load of 1 lb. at a cause the point, a, to take a new position, a', and the point, c, a new position at c', as shown in Fig. 62f.

Then d_1 = vertical deflection of a due to a vertical load of unity at a;

d_2 = vertical deflection of c due to a vertical load of unity at a;

d_9 = horizontal deflection of a due to a vertical load of unity at a;

d_{10} = horizontal deflection of c due to a vertical load of unity at a.

Let a vertical load of 1 lb. at c cause the point, a, to take a new position, a', and the point, c, a new position at c', as shown in Fig. 62g.

Then d_3 = vertical deflection of a due to a vertical load of unity at c;

d_4 = vertical deflection of c due to a vertical load of unity at c;

d_{11} = horizontal deflection of a due to a vertical load of unity at c;

d_{12} = horizontal deflection of c due to a vertical load of unity at c.

Let a horizontal load of unity at a, as shown in Fig. 62h, cause the points, a and c, to take the new positions, a' and c', respectively.

Then d_5 = horizontal deflection of a due to a horizontal load of unity at a;

d_6 = horizontal deflection of c due to a horizontal load of unity at a;

d_{13} = vertical deflection of a due to a horizontal load of unity at a;

d_{14} = vertical deflection of c due to a horizontal load of unity at a.

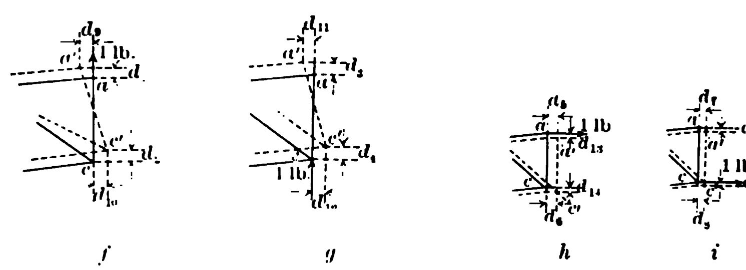


FIG. 62.

Let a horizontal load of unity at c cause the points, a and c, to take the new positions, a' and c', respectively, as shown in Fig. 62i.

Then d_7 = horizontal deflection of a due to a horizontal load of unity at c;

d_8 = horizontal deflection of c due to a horizontal load of unity at c;

d_{15} = vertical deflection of a due to a horizontal load of unity at c;

d_{16} = vertical deflection of c due to a horizontal load of unity at c.

The four simultaneous equations will now be written, from which the four unknown forces, V_a, V_c, H_a, and H_c, acting at the crown and due to a load of unity at any point, x, may be determined.

For the points, a and b, and the points, c and d, connected, under a vertical downward load at x, the motion of a must equal the motion of b, and the motion of c must equal the motion of d.

The downward motion of $a = \Delta_1 - V_a d_1 - V_c d_3 + H_a d_{13} + H_c d_{15}$,

and the downward motion of $b = +V_a d_1 + V_c d_3 + H_a d_{13} + H_c d_{15}$,

and these must be equal, as they are for the same point.

Therefore,

$$\Delta_1 - V_a d_1 - V_c d_3 + H_a d_{13} + H_c d_{15} = V_a d_1 + V_c d_3 + H_a d_{13} + H_c d_{15},$$

and simplifying,

$$V_a d_1 + V_c d_3 = \frac{\Delta_1}{2}. \qquad (1)$$

The downward motion of $c = \Delta_2 - V_a d_2 - V_c d_4 + H_a d_{14} + H_c d_{16}$,
and the downward motion of $d = +V_a d_2 + V_c d_4 + H_a d_{14} + H_c d_{16}$,
therefore,

$$\Delta_2 - V_a d_2 - V_c d_4 + H_a d_{14} + H_c d_{16} = {}_aVd_2 + V_c d_4 + H_a d_{14} + H_c d_{16},$$

and

$$V_c d_2 + V_c d_4 = \frac{\Delta_2}{2}. \qquad (2)$$

The motion to the right of $a = \Delta_3 - V_a d_9 - V_c d_{11} + H_a d_5 + H_c d_7$,
and the motion to the right of $b = -V_a d_9 - V_c d_{11} - H_a d_5 - H_c d_7$,
therefore,

$$\Delta_3 - V_a d_9 - V_c d_{11} + H_a d_5 + H_c d_7 = -V_a d_9 - V_c d_{11} - H_c d_5 - H_c d_7,$$

and

$$H_a d_5 + H\, d_7 = -\frac{\Delta_3}{2}. \qquad (3)$$

The motion to the right of $c = -\Delta_4 + V_a d_{10} + V_c d_{12} + H_a d_6 + H_c d_8$,
and the motion to the right of $d = +V_c d_{10} + V_c d_{12} - H_a d_6 - H_c d_8$,
therefore,

$$-\Delta_4 + V_c d_{10} + V_c d_{12} + H_a d_6 + H_c d_8 = V_a d_{10} + V_c d_{12} - H_a d_6 - H_c d_8,$$

and

$$H_a d_6 + H_c d_8 = \frac{\Delta_4}{2}. \qquad (4)$$

Solving these four equations, the following values for the unknown forces are found, in terms of twelve easily determined deflections:

$$V_a = +\Delta_1 \frac{d_4}{2(d_1 d_4 - d_2 d_3)} - \Delta_2 \frac{d_3}{2(d_1 d_4 - d_2 d_3)};$$

$$V_c = -\Delta_1 \frac{d_2}{2(d_1 d_4 - d_2 d_3)} + \Delta_2 \frac{d_1}{2(d_1 d_4 - d_2 d_3)};$$

$$H_a = -\Delta_3 \frac{d_8}{2(d_5 d_8 - d_6 d_7)} - \Delta_4 \frac{d_7}{2(d_5 d_8 - d_6 d_7)};$$

$$H_c = +\Delta_3 \frac{d_6}{2(d_5 d_8 - d_6 d_7)} + \Delta_4 \frac{\Delta_5}{2(d_5 d_8 - d_6 d_7)}.$$

If the values of Δ_1 to Δ_4, and d_1 to d_8, as found by actual computation, are in the direction heretofore assumed, their numerical value is to be inserted in the foregoing expressions without regard to sign, if they are found to be in a direction opposite to that assumed, they must be inserted with a minus sign.

From Maxwell's Theorem, it is known that:

$$d_2 = d_3, \quad \text{and} \quad d_6 = d_7.$$

For any particular configuration of members in the arch rib at the crown very much simpler expressions for V_a and V_c may be written.

For the case assumed, to illustrate this method, a load at any panel point other than a, makes

$$\Delta_1 = \Delta_2, \quad \text{and} \quad d_3 = d_4, \quad \text{and therefore} \quad V_a = 0.$$

A load of unity at panel point, a, makes

$$\Delta_1 = d_1, \quad \Delta_2 = d_2, \quad \text{and} \quad d_3 \text{ is still} = d_4, \text{ therefore} \quad V_a = \tfrac{1}{2}.$$

For any panel point of the rib loaded except a

$$V_c = \frac{\Delta_2}{2d_4}.$$

And for panel point, a, loaded:

$$V_c = 0.$$

Restated, for any panel point loaded other than a

$$V_a = 0, \quad \text{and} \quad V_c = \frac{\Delta_2}{2d_4}.$$

And for panel point, a, loaded:

$$V_a = \tfrac{1}{2}, \quad \text{and} \quad V_c = 0.$$

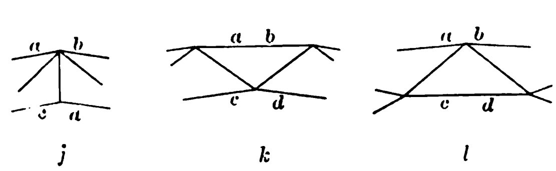


FIG. 62.

For such a configuration at the crown as indicated in Fig. 62j, that is, where the diagonals adjacent to the crown meet at the top of the crown vertical, $d_1 = d_2$, and d_3 and d_4 are no longer equal.

Then, for any panel point loaded other than c,

$$\Delta_1 = \Delta_2,$$

and therefore,

$$V_a = \frac{J_1}{2d_1}, \quad \text{and} \quad V_c = 0.$$

And for panel point c, loaded,

$$J_1 = J_2 - (d_4 - d_3), \quad \text{and} \quad J_2 = d_4,$$

and therefore,

$$V\ = 0, \quad \text{and} \quad V_c = \tfrac{1}{2}.$$

For such an arrangement of members at the crown as shown in Fig. 62k, it is readily observed that

$$V_a = 0, \quad \text{and} \quad V_c = \frac{J_2}{2d_4},$$

for any panel point loaded.

For such an arrangement at the crown as shown in Fig. 62l, it is seen that

$$V_a = \frac{J_1}{2d_1}, \quad \text{and} \quad V_c = 0,$$

for any panel point loaded.

The expressions for the horizontal crown forces remain the same for all the cases selected.

Letting $2(d_5d_8 - d_6^2) = n$, and remembering that $d_6 = d_7$, we have, for the value of the horizontal thrusts,

$$H_a = -J_3\frac{d_8}{n} - J_4\frac{d_6}{n},$$

and

$$H_c = +J_3\frac{d_6}{n} + J_4\frac{d_5}{n}.$$

The above expressions for the unknown crown forces have been written from assumed values of the twelve deflections of the half arch acting as a statically determined structure. For any actual problem, these deflections, of course, should be determined accurately, both as to direction and magnitude. The expressions for the value of the unknown crown forces will always be of the same general form as the above, but may differ in some of the signs of the terms.

Resulting plus values for V_a, V_c, H_a, H_c, indicate that they act as assumed, and minus values that they act in the opposite direction.

PROBLEM

62a. Derive expressions for the unknown forces at c and d of Fig. 62b and 62c, when the two members which meet at each of the points a and b are considered removed, and the two figures act as one truss for a load at any point on the left half.

ART. 63. THE OPEN-WEBBED ARCH WITHOUT HINGES. TEMPERATURE STRESSES

Let Fig. 63*a* show the motions of *a* and *b*, and *c* and *d*, for a rise in temperature when each half of the arch is considered as a separate structure, the full lines representing the normal figure and the dotted lines the figure changed by a rise in temperature.

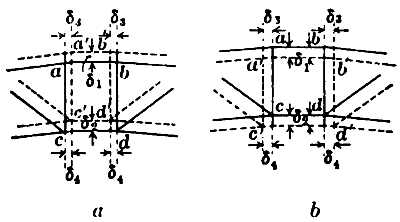


FIG. 63.

Let Fig. 63*b* show the motions of *a* and *b*, and *c* and *d*, for a corresponding fall in temperature under the same conception as before.

Then δ_1 = vertical deflection of *a* and *b* due to a certain change in temperature;
δ_2 = vertical deflection of *c* and *d* due to a certain change in temperature;
δ_3 = horizontal deflection of *a* and *b* due to a certain change in temperature;
and δ_4 = horizontal deflection of *c* and *d* due to a certain change in temperature.

Four equations for the unknown crown forces due to connecting the two halves may now be written for both a rise and a fall in temperature, using the notation as previously defined.

For a rise in temperature:

The upward motion of $a = +\delta_1 + V_a d_1 + V_c d_3 - H_a d_{13} - H_c d_{15}$.

and the upward motion of $b = +\delta_1 - V_a d_1 - V_c d_3 - H_a d_{13} - H_c d_{15}$.

For the two halves connected,

these motions must be equal; therefore,

$$V_a d_1 + V_c d_3 = 0. \qquad (5)$$

In an entirely similar manner, the three following equations may be written:

$$V_a d_2 + V_c d_4 = 0, \quad \ldots \quad (6)$$

$$H_a d_5 + H_c d_7 = -\partial_3, \quad \ldots \quad (7)$$

$$H_a d_6 + H_c d_8 = -\partial_4. \quad \ldots \quad (8)$$

For a fall in temperature:

$$V_a d_1 + V_c d_3 = 0, \quad \ldots \quad (5a)$$

$$V_a d_2 + V_c d_3 = 0, \quad \ldots \quad (6a)$$

$$H_a d_5 + H_c d_7 = \partial_3, \quad \ldots \quad (7a)$$

$$H_a d_6 + H_c d_8 = \partial_4. \quad \ldots \quad (8a)$$

An examination of Eqs. (5), (6), (5a), and (6a), shows that V_a and V_c are zero; therefore, the only crown forces due to temperature are horizontal.

For a rise in temperature, remembering that $d_6 = d_7$, and making $(d_5 d_8 - d_6^2) = g$:

$$H_a = -\partial_3 \frac{d_8}{(d_5 d_8 - d_6^2)} + \partial_4 \frac{d_6}{(d_5 d_8 - d_6^2)} = -\partial_3 \frac{d_8}{g} + \partial_4 \frac{d_5}{g},$$

and

$$H_c = -\partial_3 \frac{d_6}{(d_5 d_8 - d_6^2)} + \partial_4 \frac{d_5}{(d_5 d_8 - d_6^2)} = -\partial_3 \frac{d_6}{g} + \partial_4 \frac{d_5}{g}.$$

For a fall in temperature,

$$H_a = +\partial_3 \frac{d_8}{(d_5 d_8 - d_6^2)} - \partial_4 \frac{d_6}{(d_5 d_8 - d_6^2)} = +\partial_3 \frac{d_8}{g} - \partial_4 \frac{d_6}{g},$$

and

$$H_c = +\partial_3 \frac{d_6}{(d_5 d_8 - d_6^2)} - \partial_4 \frac{d_5}{(d_5 d_8 - d_6^2)} = \partial_3 \frac{d_6}{g} - \partial_4 \frac{d_5}{g}.$$

The foregoing gives a simple method for finding the stresses in every member of the arch due to a load at any point, or for any desired change in temperature. Under this method, the work may be easily divided into several sets of independent and easily understood operations at the very beginning of the computations, thereby leaving only the small and easily managed theoretical part for the person in charge.

PROBLEMS

No. **63***a*. If the distance between the points of support at each end of the arch of Fig. 62*a* changes according to the same law as that for the change in length of the various rib members, how may ∂_1, ∂_2, ∂_3, and ∂_4 as defined in this article be determined?

No. **63** *b*. If the distance between the points of support at each end of the arch of Fig. 62*a* remains constant, how may ∂_1, ∂_2, ∂_3, and ∂_4, as defined in this article be determined?

CHAPTER IX

DISTORTIONS OF STRUCTURES AS AFFECTING THEIR ERECTION

ART. 64. CAMBER

In course of construction of all bridges of any magnitude, a knowledge of the relation of the shape of the unstressed or partially stressed figure to that of the figure for which the stresses were originally computed and to the figure under full load is necessary in order that the various parts may be properly assembled. It is also highly important to know in advance the final shape of a structure under its loading, as any considerable change in form from that contemplated in the design might give the structure an undesirable appearance.

With a knowledge of the elastic action of a structure it is possible to give it under a given loading any desired form, or camber, as it is termed in practice, and having fixed the lengths of the various members of the structure to produce this form, its form without load or with any other state of loading may be found. This knowledge, as can readily be seen, is absolutely essential to the successful erection of the structure.

For an example of the method which may be used to secure a desired form let it be required to adjust the lengths of the members of the truss of Art. 44 in such a manner that the bottom chord will be horizontal under its own weight and a full live load.

The truss for the purpose of this problem will be considered as pin connected and all joints will be made with full pin bearings. Tension members will be lengthened by the amount of the necessary play in the pin holes and compression members shortened by the same amount as has been shown in Art. 45. The play in the pin holes will be taken as 1/32 in. = .0026 ft. The lengths of all members should be computed to five decimal places when expressed in feet and the length taken to the nearest four places. These lengths will be called the normal lengths. The lengths corrected to give panel points of the bottom chord on a horizontal line will be called the manufactured length l_m. It will be noted that members which are lengthened when under stress must be manufactured shorter by the amount of such lengthening, and the converse for members which are shortened under stress.

The computations for this problem will be made analytically and may be arranged as shown in the table of page 218.

The notation at the head of the various columns in the table is defined in the following:

l_n = the normal length of the member;

l_m = the length to which it would be desirable to manufacture the member;

S, A and E are as previously defined;

e_1 = elastic change in length of the member for its manufactured length, and this may be taken as the same for l_n without appreciable error;

e_2 = play in pin holes, which should never exceed $\frac{1}{32}$ in.;

$e = e_1 + e_2$;

$l_m = l_n \pm e$, in feet expressed to the nearest fourth decimal figure;

$l_m'' = l_m$ expressed in feet and inches to the nearest $\frac{1}{64}$ in.;

l_{mc}'' = corrected manufactured length, which is the same in this problem as l_m'' except for the two inclined end posts;

$l_m' = l_m''$ expressed in feet to the nearest fourth decimal figure;

$e_3 = l_m' - l_m$ = unavoidable errors in the length of the members due to the incommensurable units;

T_4 = stress due to a vertical load of unity at point No. 4;

δ_4 = partial deflection at point No. 4 due to e_3 in any member;

$\Sigma\delta_4$ = total deflection at point No. 4 due to e_3 in all members;

c = final correction in inches to be appiled to a few members to bring the deflection to zero at the desired point, in this case point No. 4.

In order that the previous notation may be better understood and the reason for its adoption made dear, it may be said that:

It is necessary for American and English shop methods to express l_m in the duodecimal system. This leads to adopting a length of member almost always a little longer or shorter than desired, owing to the incommensurability of the units of the decimal and duodecimal systems, which gives a shape of structure slightly different from that required. The deviation from a desired form in small bridges is a trivial matter, but in long spans and in cantilever and drawbridges it is often very important.

It is impossible by the best American shop methods to work to a smaller unit than $\frac{1}{64}$ in. = .001302 ft. +, = .0013 ft. +. There may then be a difference eitherway of as much as .0006 ft. in the manufactured length from that desired. The influenceof this difference in producing deflection at certain points may be many times this

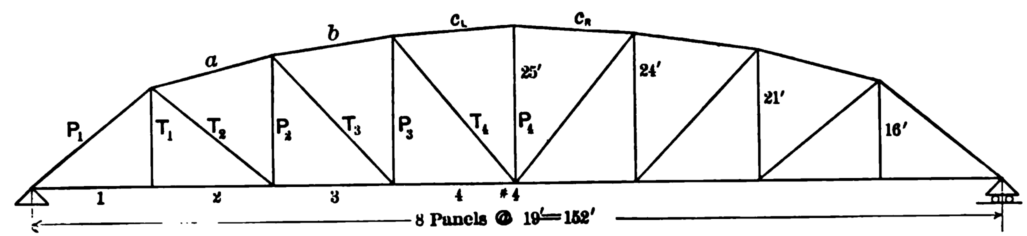


FIG. 64a.

TABLE No. 64a

	l_n	Stress Live+Dead. S	Sq. Ins. A	e_1	e_2	e_1+e_2 e	$l_n \pm e$ l_m	l_m''	l_m'	e_3	T	∂_4
P_1	24.8395	−134100	16.72	−.0067	−.0026	−.0093	24.8487	24'−10$\frac{12}{64}$"	24.8490	+.0003	−0.776	−.0002
P_2	21.0000	− 3700	5.29	−.0005	−.0026	−.0031	21.0031	21'− 0$\frac{2}{64}$"	21.0026	−.0005	−0.262	+.0001
P_3	24.0000	+ 8900	5.29	+.0013	+.0026	+.0039	23.9961	23'−11$\frac{61}{64}$"	23.9961	0	−0.312	0
P_4	25.0000	+ 13900	5.29	+.0022	+.0 26	+.0048	24.9952	24'−11$\frac{60}{64}$"	24.9948	−.0004	+0.160	−.0000*
T_1	16.0000	+ 22800	5.29	+.0023	+.0026	+.0049	15.9951	15'−11$\frac{60}{64}$"	15.9948	−.0003	0	0
T_2	24.8395	+ 41000	3.75	+.0090	+.0026	+.0116	24.8278	24'− 9$\frac{60}{64}$"	24.8281	+.0003	+0.407	+.0001
T_3	28.3196	+ 18700	7.06	+.0025	+.0026	+.0051	28.3145	28'− 3$\frac{50}{64}$"	28.3151	+.0006	+0.421	+.0003
T_4	30.6105	+ 5800	7.06	+.0008	+.0026	+.0034	30.6070	30'− 7$\frac{18}{64}$"	30.6068	−.0002	+0.536	−.0001
a	19.6469	−139000	13.89	−.0065	−.0026	−.0091	19.6560	19'− 7$\frac{56}{64}$"	19.6562	+.0002	−0.936	−.0002
b	19.2354	−148200	13.89	−.0068	−.0026	−.0094	19.2448	19'− 2$\frac{60}{64}$"	19.2448	0	−1.2 2	0
c_l	19.0263	−150300	13.89	−.0069	−.0026	−.0095	19.0358	19'− 0$\frac{27}{64}$"	19.0352	−.0006	−1.522	+.0009
1	19.0000	+102600	4.50	+.0144	+.0026	+.0170	18.9830	18'−11$\frac{51}{64}$"	18.9831	+.0001	+0.594	+.0001
2	19.0000	+102600	4.50	+.0144	+.0026	+.0170	18.9830	18'−11$\frac{51}{64}$"	18.9831	+.0001	+0.594	+.0001
3	19.0000	+134000	8.75	+.0097	+.0026	+.0123	18.9877	18'−11$\frac{55}{64}$"	18.9883	+.0006	+0.905	+.0005
4	19.0000	+146600	9.37	+.0099	+.0026	+.0125	18.9875	18'−11$\frac{54}{64}$"	18.9870	−.0005	+1.188	−.0006

$c_{P_1} = \frac{+.0010}{-.776} = -.0013 = \frac{1}{64}''$ $\therefore\ l_{mc}'' = 24' - 10\frac{13}{64}''$ and $l_m' = 24.8505'$. $c_{c_l} = \frac{+.0010}{-.1522} - .00066'$. $\Sigma = +.0010$.

* = ½ of ∂_4.

amount, as can readily be seen by multiplying the difference by the stress in the member due to a load of unity acting at the point whose deflection is desired.

It will be sufficiently exact for the problem here selected to correct the length of the member or members contributing most to the deflection of the center bottom chord point for the error introduced by the incommensurability of the unity by a sufficient amount to give this chord point a deflection of zero. If the members could be made of a length l_m, the truss would have zero deflection.

The deflection of the center bottom chord point using the manufactured lengths l'_m stated is 2(.0010). The two center top chords are the most influential in producing deflection at this point, and in general both should be corrected the same amount so that c_r and c_l may be manufactured alike. In order to make this deflection zero let c = the required correction to c_l and c_r, then $2cT = .0010 \times 2$ or $c = \frac{.00105}{1.522} = .00066$ ft. which is too small an amount to be applied to c_r and c_l. For this truss the correction should be applied to the two end posts, the correction $c = \frac{.0010}{.776} = .0013$ and $l'_m = 24.8505$.

ART. 65. CAMBER BLOCKING, ANALYTICAL METHOD

Having arrived at the desired lengths to be used in manufacture to give zero deflection for the structure fully loaded for the center bottom chord panel point and practically zero deflection for the other bottom chord points, we must determine the location of these same points for no stress in the members, in order that the pins, or rivets for a riveted truss, at the joints may be driven when the structure is erected.

It is evident that if the erection of this bridge is attempted by simply placing it on a series of points at the same elevation that the connections may be very difficult or even impossible and that if connection is possible a very unequal loading may result on the falsework, and injury to the truss members by thus forcing them together. The tabulated computation for the proper amount of Camber Blocking follows. In order to make the table clear the following notation is defined:

Let Δ_1, Δ_2, Δ_3 and Δ_4 = the required elevation of points number 1, 2, 3 and 4 respectively, when the members are assembled and without stress;

T_1, T_2, T_3 and T_4 = stresses in the members of the truss caused by a vertical load of unity at points number 1, 2, 3 and 4 respectively;

$e_4 = l_{mc} - l_n$, in which l_{mc} = manufactured length c. to c., and l_n = normal length c. to c.

Then an analytical computation of Δ_1, Δ_2, Δ_3 and Δ_4 may be made as shown by Table No. 65*a*, of this page. It is seen from this table that $\Delta_1 = \Sigma T_1 e_4$, $\Delta_2 = \Sigma T_2 e_4$, $\Delta_3 = \Sigma T_3 e_4$, and $\Delta_4 = \Sigma T_4 e_4$. Therefore, points Nos. 1, 2, 3 and 4 must be blocked up, $1.17'' = 1\frac{3}{16}''$, $1.83'' = 1\frac{13}{16}''$, $2.19'' = 2\frac{3}{16}''$ and $2.31'' = 2\frac{5}{16}''$, respectively, in order that the pins may be easily driven and that the falsework be loaded properly at all points.

It should be noted here that if the falsework bents are high in any case the change in the lengths of the posts in the bents should be added to the amount of camber blocking determined from a consideration of e_4 alone.

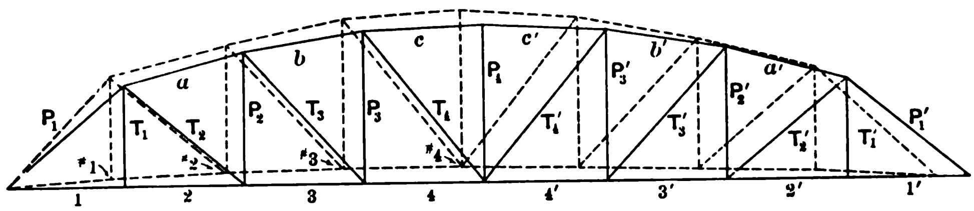


FIG. 65*a*.

TABLE No. 65*a*

	e_4	T_1	T_2	T_3	T_4	T_1e_4	T_2e_4	T_3e_4	T_4e_4
P_1	+.0108	−1.358	−1.164	−0.970	−0.776	−.0147	−.0126	−.0105	−.0084
P_2	+.0026	+0.302	+0.627	−0.328	−0.262	+.0008	+.0016	−.0009	−.0007
P_3	−.0039	+0.203	+0.406	−0.610	−0.312	−.0008	−.0016	+.0024	+.0012
P_4	−.0052	+0.040	+0.080	+0.120	+0.160	−.0002	−.0004	−.0006	−.0008
T_1	−.0052	+1.000	0	0	0	−.0052	0	0	0
T_2	−.0114	−0.469	+0.611	+0.509	+0.407	+.0053	−.0070	−.0058	−.0046
T_3	−.0045	−0.274	−0.547	+0.526	+0.421	+.0012	+.0025	−.0024	−.0019
T_4	−.0037	−0.185	−0.370	−0.434	+0.536	+.0007	+.0014	+.0016	−.0020
a	+.0093	−0.700	−1.413	−1.169	−0.936	−.0065	−.0131	−.0109	−.0087
b	+.0094	−0.500	−1.002	−1.502	−1.202	−.0047	−.0094	−.0141	−.0113
c	+.0089	−0.381	−0.761	−1.141	−1.522	−.0003	−.0068	−.0102	−.0135
1	−.0169	+1.038	+0.890	+0.742	+0.594	−.0175	−.0150	−.0125	−.0100
2	−.0169	+1.038	+0.890	+0.742	+0.594	−.0175	−.0150	−.0125	−.0100
3	−.0117	+0.679	+1.357	+1.131	+0.905	−.0079	−.0159	−.0132	−.0106
4	−.0130	+0.495	+0.990	+1.484	+1.187	−.0064	−.0129	−.0193	−.0154
P'_1	+.0108	−0.194	−0.388	−0.582	−0.776	−.0021	−.0042	−.0063	−.0084
P'_2	+.0026	−0.066	−0.131	−0.197	−0.262	−.0002	−.0003	−.0005	−.0007
P'_3	−.0039	−0.078	−0.157	−0.234	−0.312	+.0003	+.0006	+.0009	+.0012
T'_1	−.0052	0	0	0	0	0	0	0	0
T'_2	−.0114	+0.102	+0.203	+0.305	+0.407	−.0012	−.0023	−.0035	−.0046
T'_3	−.0045	+0.105	+0.212	+0.316	+0.421	−.0005	−.0010	−.0014	−.0019
T'_4	−.0037	+0.134	+0.268	+0.401	+0.536	−.0005	−.0010	−.0015	−.0020
a'	+.0093	−0.234	−0.467	−0.702	−0.936	−.0022	−.0043	−.0065	−.0087
b'	+.0094	−0.300	−0.601	−0.901	−1.202	−.0028	−.0056	−.0085	−.0113
c'	+.0089	−0.381	−0.761	−1.141	−1.522	−.0034	−.0068	−.0102	−.0135
1′	−.0169	+0.148	+0.297	+0.445	+0.594	−.0025	−.0050	−.0075	−.0100
2′	−.0169	+0.148	+0.297	+0.445	+0.594	−.0025	−.0050	−.0075	−.0100
3′	−.0117	+0.226	+0.452	+0.678	+0.905	−.0026	−.0053	−.0079	−.0106
4′	−.0130	+0.297	+0.594	+0.890	+1.187	−.0039	−.0077	−.0116	−.0154
					Σ ft. =	−.0978	−.1521	−.1809	−.1926
					Σ ins. =	1.1736	1.8253	2.1908	2.3112

ART. 66. CAMBER BLOCKING, GRAPHICAL METHOD

The analytical computation of the elevations of the panel points for the truss without stress requires the computation of T_1, T_2, T_3 and T_4, which takes considerable time. For problems of this nature where a high degree of accuracy is not

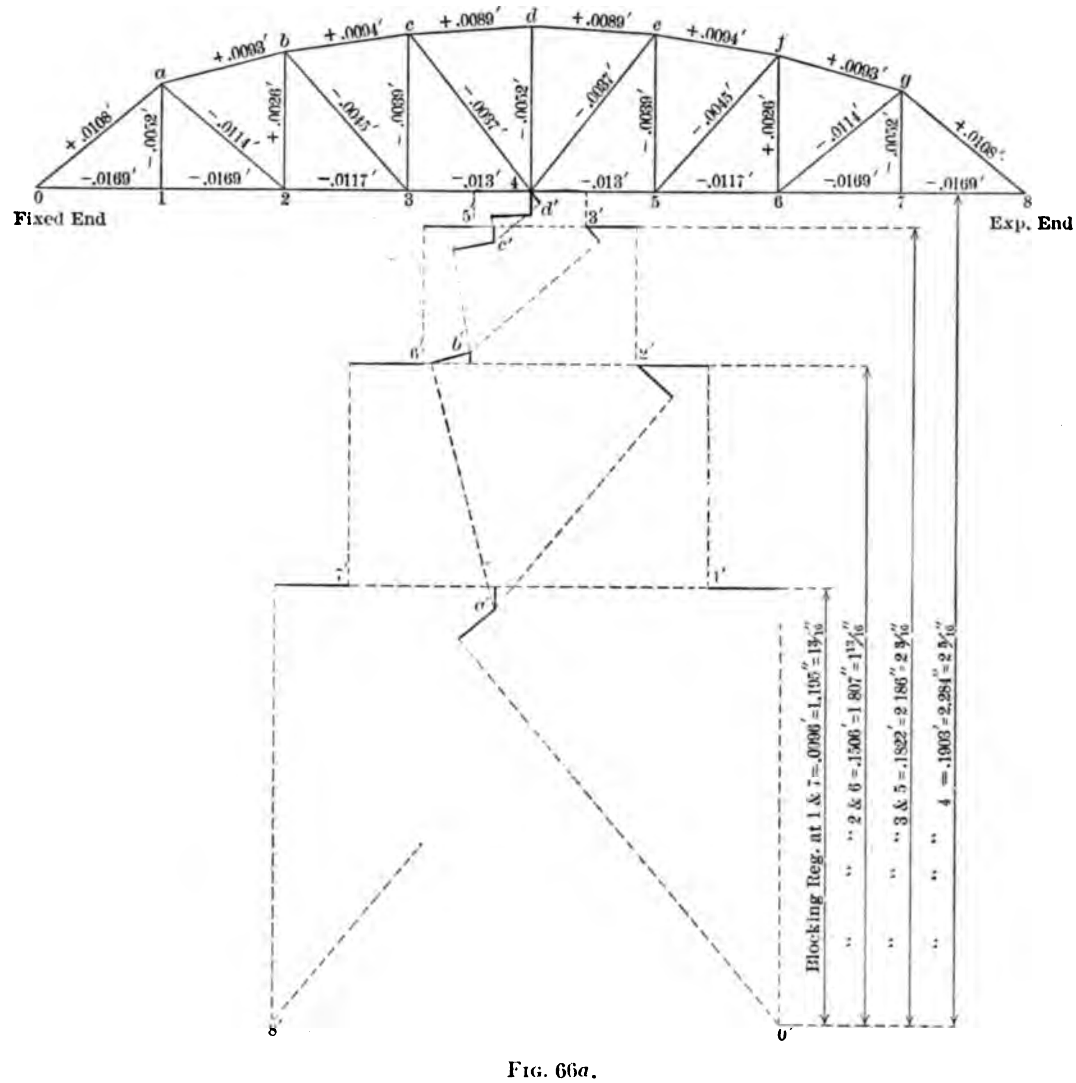


FIG. 66a.

necessary the graphical method gives a very good solution, by using the changes in length e_4 as indicated on Fig. 66a, which are those of Table No. 65a. A deflection diagram can be constructed, as has been explained in Art. 46. The diagram is that of Fig. 66a.

The horizontal motion of any panel point of the bottom chord is the sum of the changes in length of the members of the bottom chord between the fixed end and the point in question, and is given by this method by scale if desired; but these motions are more readily obtained by summing up the changes arithmetically.

The horizontal motions are:

For point	No. 1	$.0169' = \frac{13}{64}''$	to the left.
"	No. 2	$.0338' = \frac{13}{32}''$	"
"	No. 3	$.0455' = \frac{35}{64}''$	"
"	No. 4	$.0585' = \frac{45}{64}''$	"
"	No. 5	$.0715' = \frac{55}{64}''$	"
"	No. 6	$.0832' = 1''$	"
"	No. 7	$.1001' = 1\frac{13}{64}''$	"
"	No. 8	$.1170' = 1\frac{13}{32}''$	"

In removing the camber blocking after the span has been erected it will be well to begin at the right or roller end and take it out panel point by panel point, going to the left or fixed end. The vertical deflections are:

For points	No. 1 and 7	up	$1\frac{3}{16}''$
"	No. 2 and 6	"	$1\frac{13}{16}''$
"	No. 3 and 5	"	$2\frac{3}{16}''$
and for point	No. 4	"	$2\frac{5}{16}''$

Which values for the vertical deflections agree very closely with those computed analytically and are close enough for the purpose of determining the amount of camber blocking.

This chapter might be extended indefinitely, as it is possible to illustrate it by many most interesting examples. Enough has been given, however, to enable the engineer to undertake any problem which arises in the erection of a framed truss.

In the following chapter the methods which have been successfully employed for the control and regulation of certain deflections of important bridges are quite fully described.

CHAPTER X

ADJUSTING DEVICES AND THE NECESSARY ADJUSTMENTS REQUIRED TO MAKE THE FINAL CONNECTION FOR IMPORTANT STRUCTURES

ART. 67. INTRODUCTORY

THE previous chapter has shown that successful bridge construction demands a recognition of the fact that the elastic properties of the materials must be understood and considered at the proper stage of the design and construction. In the previous chapter only such bridges were considered as could be erected on a temporary structure, or falsework, the function of the falsework being to carry the permanent structure until it becomes self-supporting. In many locations the placing of any such falsework is impossible for one or more of many reasons. The requirements of navigation, great depth of water, poor bearing power of the soil below the structure, or a very swift current are often such as to prohibit the use of falsework in erecting a permanent work.

Very frequently erection conditions determine the type of structure to be used in a given location or govern certain features of the design to enable the erection conditions to be met.

Any structure which has an outline which is fixed by the requirements for permanent use and which is under a different state of stress during erection from that when in permanent use requires special consideration of the elastic deformations under all conditions of stress in order that its construction may be possible at all stages of its manufacture.

Arched bridges are frequently erected as cantilevers from each abutment; during erection such structures would have the top chord of the arch rib in tension and the bottom chord in compression. While for use during service both the top and bottom chords would in general be in compression. It is clear that for this case the connection at the center during erection cannot be made unless special precautions are taken.

Very often simple span bridges are erected as cantilevers, here also the different stress conditions during erection from that of the structure in final use must be recognized.

In the last article of this chapter a method of making the final connection for an arch and a simple span which were erected as cantilevers will be described.

Cantilever bridges are designed to carry loads in a definite manner and to have a certain form under a given loading when completed and ready for service. The cantilever principle may be applied in a great variety of ways with which the reader will be assumed to be familiar. The erection of the structure is subject to little difficulty provided forethought is exercised and the proper manner of meeting each condition as it will develop is carefully worked out in advance of construction.

In order to study the manner of erecting this type of bridge the location will be assumed to be such as to demand a structure of the form shown in Fig. 67*a*.

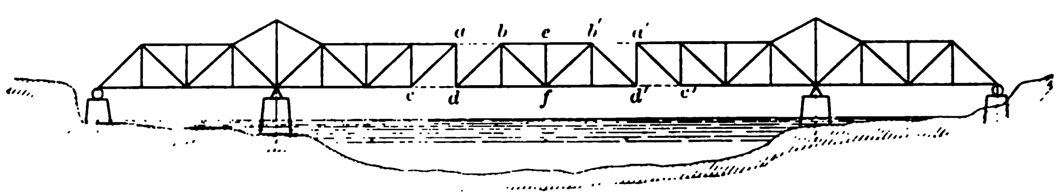


FIG. 67*a*.

The general order of the erection for such a bridge is as follows:

I. The anchor arms on falsework, the bents of which are under each panel point of the structure.

II. The cantilever arms and suspended span acting as two cantilevers meeting at the center of the crossing.

The stresses for these conditions are not those which will exist for the bridge when completed. The members *ab*, *a'b'*, *cd* and *c'd'* of Fig. 67*a* do not carry stress for the bridge completed, while during erection thay have considerable stress, the amount of course depending on the size of the structure and the type of traveler used in erection.

The top chord *bb'* of the suspended span is under compression in the final structure and under tension during erection.

The bottom chord *dd'* is in tension during use and in compression during erection.

In the following, Art. 68, will be described the adjusting devices used in making the final connection during erection for two cantilevers of widely different magnitude.

In Arts. 69, 70 and 71 other erection devices for making adjustments in erecting cantilevers, arches, and simple spans will be shown and described.

It should be remembered that while these devices successfully accomplished

the work they were designed to do and represent perhaps the best that has been done along these lines in American bridge building, nevertheless they are in a sense experimental. And the author ventures to state that the users of these devices had considerable anxiety as to the successful outcome of their employment.

Wherever possible the stresses under which the devices illustrated were used have been given with the idea that they might be of service to engineers on future work. It should be borne in mind, however, that such adjustments cannot be copied with safety and that only engineers of experience should attempt to adapt them to other use or develop new devices by their aid.

ART. 68. ERECTION DEVICES FOR CANTILEVER STRUCTURES *

In erecting structures by the cantilever method, whether these structures are permanent cantilevers or other types, such as arches or fixed spans, which are not infrequently erected as cantilevers, it is advisable to have some adjustment devices to enable the center connection to be readily made at any temperature. In the erection of some of the earlier of the important structures, notably in the Eads Arch Bridge at St. Louis, such adjusting devices to be used during erection were omitted, and the structure had to be connected by cooling one chord and heating the other, but it is much more economical and practical to use adjustment devices. To allow for temperature, deflection, and inaccuracies of shop dimensions and positions of masonry, it is necessary not only to be able to lengthen or shorten the overhanging structure, but to be able to raise or lower the end or move it horizontally in either direction.

The usual device is a screw or a toggle joint in the tension chord, and a jack or a wedge in the compression chord.

A direct tension screw can be used in very small structures only, so the toggle joint operated by a screw has become almost universal for the tension chord. For the compression chord, a hydraulic jack can be used for ordinary lengths, as such a jack has to be used only when an actual change of position has to be made, and at other times the load can be carried on fixed blocking. For very large or heavy structures, when the size of such a jack becomes impracticable, a wedge operated by a screw is generally adopted for compression chords. Fig. 68*a* shows the general outline and position of the erection devices for a small cantilever bridge of 390 ft. clear opening.

* This article was prepared by Mr. Henry W. Hodge of the firm of Boller & Hodge, Consulting Engineers, New York.

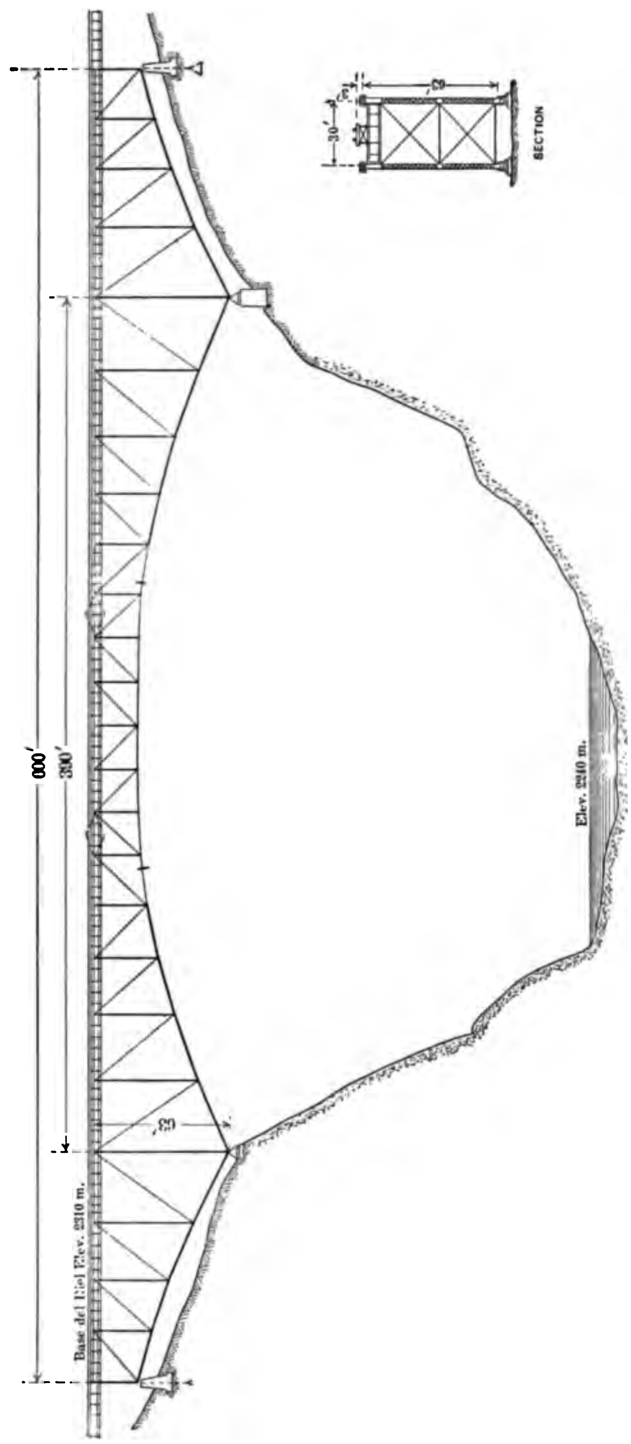


Fig. 68a.—National Lines of Mexico, Cantilever over Arroya Del Chico. March 14, 1910.
Boller & Hodge, Consulting Engineers, New York City.

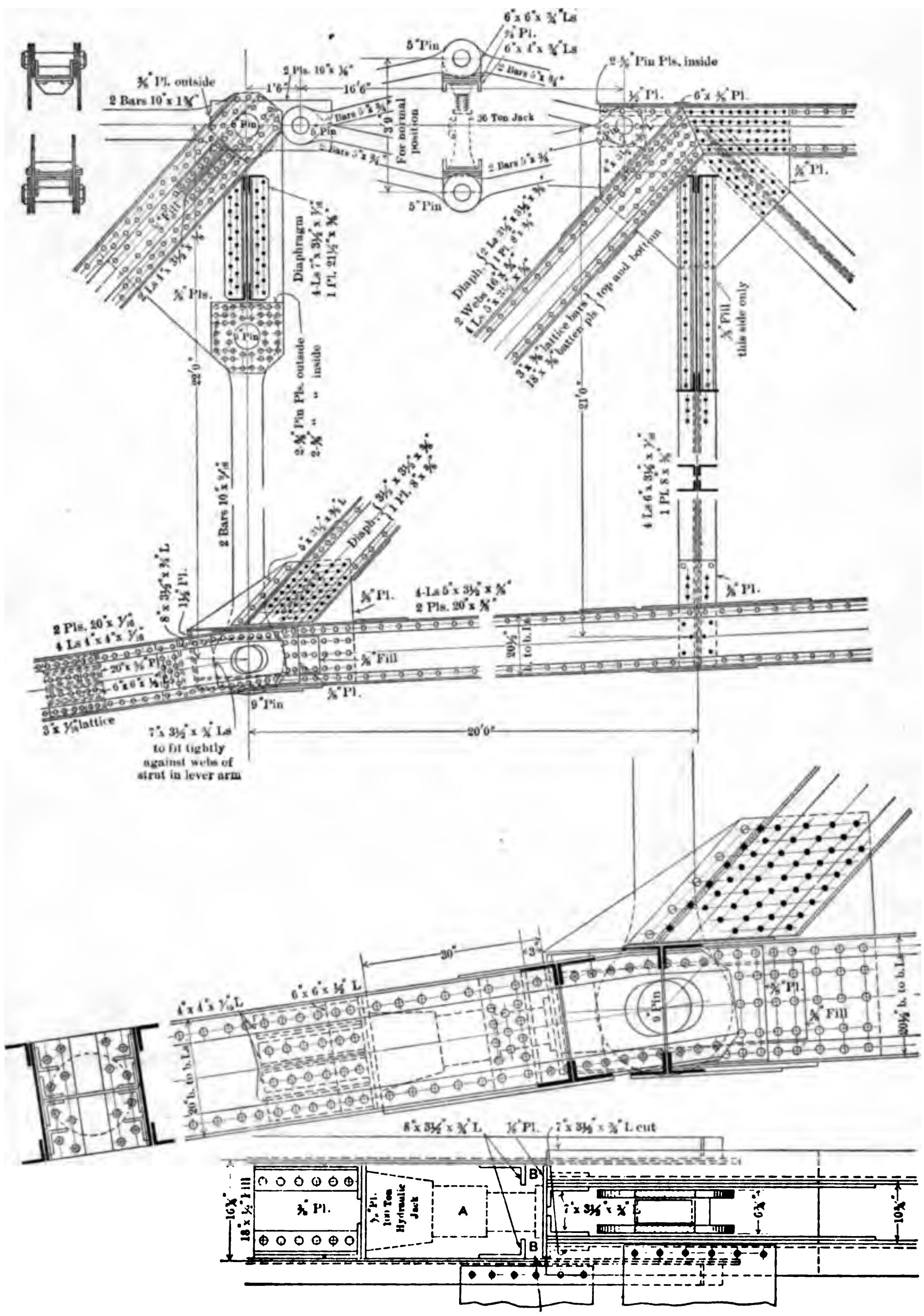


FIG. 68b.—Erection Devices Arroya Del Chico Cantilever, Mexican National Railway.
Boller & Hodge, Consulting Engineers.

In Fig. 68*b*, page 227, are shown the erection devices used in considerable detail. The maximum horizontal erection stress to be carried was 180,000 lbs.

The lower chords each had a 100-ton hydraulic jack to either lengthen or shorten them as shown at *A*. This jack had a total plunger movement of 6 ins., and the spaces shown at *B* were kept filled with a stack of thin plates, making a total thickness of about 3 ins. when the bridge was in its calculated position for final connection. These plates carried the erection stress except during actual movement of the overhanging suspended span. When the jack was put in use, these plates could either be removed or increased to a thickness of 6 ins.

The lengthening or shortening of the upper tension chord was controlled by a toggle joint, operated by a 36-ton screw jack shown at *C*. This operating jack had to carry the full erection stress at all times, so that a hydraulic jack was not practicable. This jack had a total vertical movement of 13 ins., 7 ins. above the position shown thereby, decreasing the length of panel by 1¾ ins., and 6 ins. below the position shown, increasing the panel length by 1¼ ins., thus giving a total variation in length of 3 ins. When the jack was extended to its maximum rise, the load carried by it was 24 tons, and the 36-ton jack was used to allow for overload caused by wind or impact.

It will be readily seen that these two toggles and two jacks placed at the ends of each lever arm enabled the erector to lengthen or shorten the suspended span, raise or lower it, or move it to either side, so that the connection at the center of the structure could be made under any condition.

After the span was coupled at the center, these erection devices were removed, and it will be noted that the bars forming the upper toggle joints were attached to secondary pins so that they could be readily taken out and used for any other structure.

These erection devices are always placed in the panels between the lever arm and suspended span, where there is no chord stress after the suspended span is coupled. In Fig. 68*d* is shown the general outline for the Wabash Pittsburgh cantilever over the Monongahela River, built in 1902. This is a double-track structure with a channel span of 812 ft., and the erection chord stress was 1,200,000 lbs. for each truss. In Fig. 68*c*, Plate II, are shown the erection devices used.

The toggle in the upper chord was operated by the temporary screw pulling down at panel point 19*U*. After erection this screw was removed and replaced by the permanent vertical post which supported the top chord bars. This toggle was erected in such a position that the pin 19*U* was 3 ft. 8½ ins. below its normal position, which shortened the true distance between the pins 18*U* and 20*U* by 5 ins. The stress on the temporary vertical bars 19*U*—19*M* and the operating

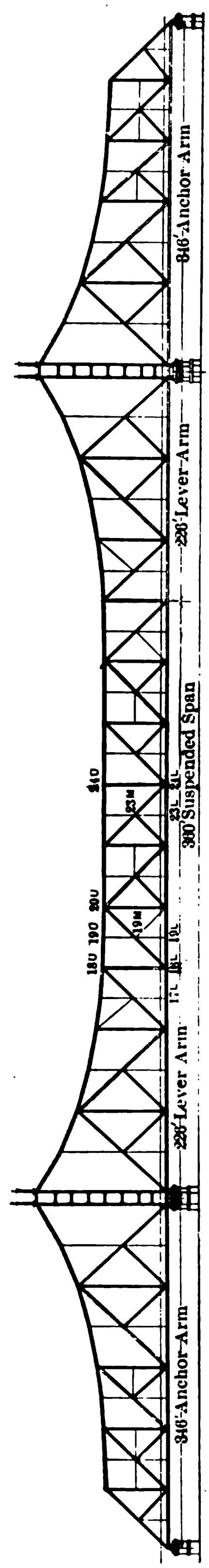


FIG. 68d.—General Elevation of the Monongahela River Cantilever.
Boller & Hodge Consulting Engineers.

screw in this position of the toggle was 300,000 lbs. The toggle was set up in this position so as to ensure having the overhanging end of the suspended span above its true position, so that it would never be necessary to exert the full power necessary to raise this overhanging end. The ratchet and screw were, however, designed of sufficient power to raise the end if necessary, but during the coupling up of the span it was only found necessary to slack off on this screw, dropping the end of the suspended span to its true position. The wedge in the lower chord was also placed in such a position that the distance between pins $17L$ and $18L$ was 5 ins. longer than the normal length, thus projecting the end of the suspended span beyond its true position. This enabled the erectors to couple up the lower chord bars in the four central panels of the suspended span, but to couple these bars the lower chord pins $23L$ were dropped below their true position, and when all the pins had been driven the wedge in the lower chord was slacked off until the chord stress came on the lower chord, thus straightening out the lower chord panels and dropping the two top chords to a butt joint at the panel point $24U$. As soon as the chords of the suspended span got their final stress, the wedge and toggle were removed, and the hangers $23M-23L$ were coupled up.

It will be noticed that the wedge worked against swinging wedge guides, which took up the varying angle of the lower chord as it dropped down to its normal position. It will also be noticed that the angle of this wedge was 1 to 5, being about the angle of the sliding friction, so that the wedge would practically squeeze out under direct compression when the screw slacked off, though this screw was designed to put a compressive force against the top of the wedge so as to force it out in case it stuck.

There were no permanent horizontal laterals in the top chord panels between $18U$ and $20U$, as the wind pressure on the upper portions of the suspended span was assumed to travel down the inclined post $18L$–$20U$. The top chord bars $18U$–$20U$ were therefore unnecessary in the permanent structure, but as it would have been difficult to remove these bars they were left in place and supported by the vertical post $19U$–$19M$.

The lower lateral stresses were carried in shear through the sliding joint in the panel $17L$–$18L$. It will be noticed that this joint had planed plates on each side so as to carry this shear from the suspended span to the lateral system of the lever arm, but this joint was free to move in a longitudinal direction, thus taking up the expansion and contraction of the suspended span as well as of the lever arm. The same amount of expansion and contraction was also allowed for in the slotted holes of the eye-bars at pin $19U$.

ART. 69. DEVICES FOR ADJUSTMENTS, IN ERECTION AND PERMANENT USE, FOR CANTILEVER BRIDGES*

The previous article gave drawings of the adjustments and a description of this operation for a cantilever with a center span of 812 ft. c. to c. of main river piers. In this article very similar devices acting under slightly higher stresses will be shown by drawings and described.

Fig. 69*a* is a general outline drawing of the cantilever bridge which carries the Pittsburgh & Lake Erie Railroad over the Ohio River at Beaver, Pa. This bridge was built in 1910 for Cooper's E_{60} Loading, and of medium steel with unit stresses about one-half the elastic limit of the material. Fig. 69*b*, Plate III, shows the devices used to make the final connections at the center. The device for controlling the length of the upper chord and raising or lowering the outer end of the cantilever consisted of a screw in the vertical member $M_{19}U_{19}$.

The device for controlling the length of the bottom chord and raising or lowering the outer end of the cantilever consisted of a wedge, having a bevel of 1 to 10 for each face, actuated by a screw and placed in member $L_{17}L_{18}$ near the L_{18} end. These devices were set so that the points U_{22}, of Fig. 69*a*, would be higher and a greater distance apart than the length of the member $U_{22}U_{22}$, and so that the points $L_{22}L_{22}$ were higher and a less distance apart than the length of the members $L_{22}L_{23}L_{22}$. The bottom chord members were connected first with the point L_{23} considerably below the line joining the L_{22} points, then the top chord closure was made by inserting the members $U_{22}U_{22}$. The closure of the top and bottom chords is clearly shown in the photograph of Fig. 69*d*. After closing the top and bottom chords the closure of the web members in the two center panels was made. The adjusting devices were then operated to change the manner of action of the suspended span from that of a cantilever to that of a simple span. This was done by lengthening the members $M_{19}U_{19}$ and shortening the members $L_{17}L_{18}$ simultaneously by means of the screws and wedges. Fig. 69*c* is a diagram showing the members containing the adjusting devices and the maximum stresses in these members during erection. In this bridge closure was made with only one erection traveler in use, as is shown in the photograph of Fig. 69*d*. This made the elastic deformations for one-half of the bridge greater than for the other. This difference in the amount of deformation was readily taken care of by the adjusting devices. Fig. 69*b* shows the amount of play in the pin holes in the members $L_{17}L_{18}$ and

* The drawings and photograph of this article were furnished by Mr. Paul L. Wolfel, Chief Engr. McClintic-Marshall Construction Company.

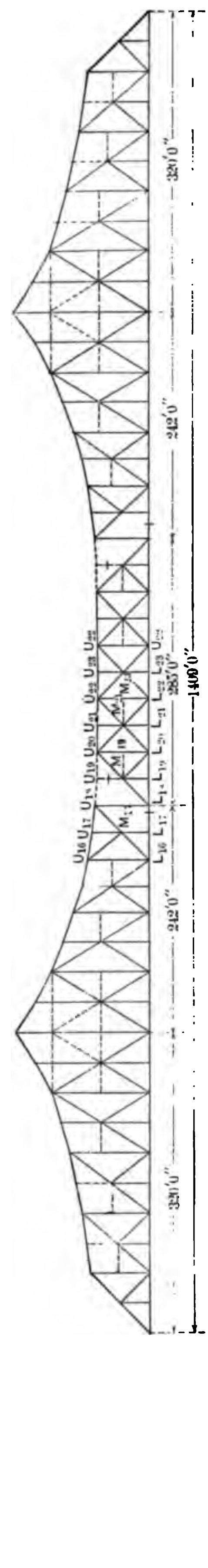


FIG. 69a.—Pittsburg and Lake Erie R.R. Cantilever Bridge over the Ohio River at Beaver, Pa.
Designed by Albert Lucius, Consulting Engineer.
Built by McClintic Marshall Construction Co., Pittsburgh, Pa.

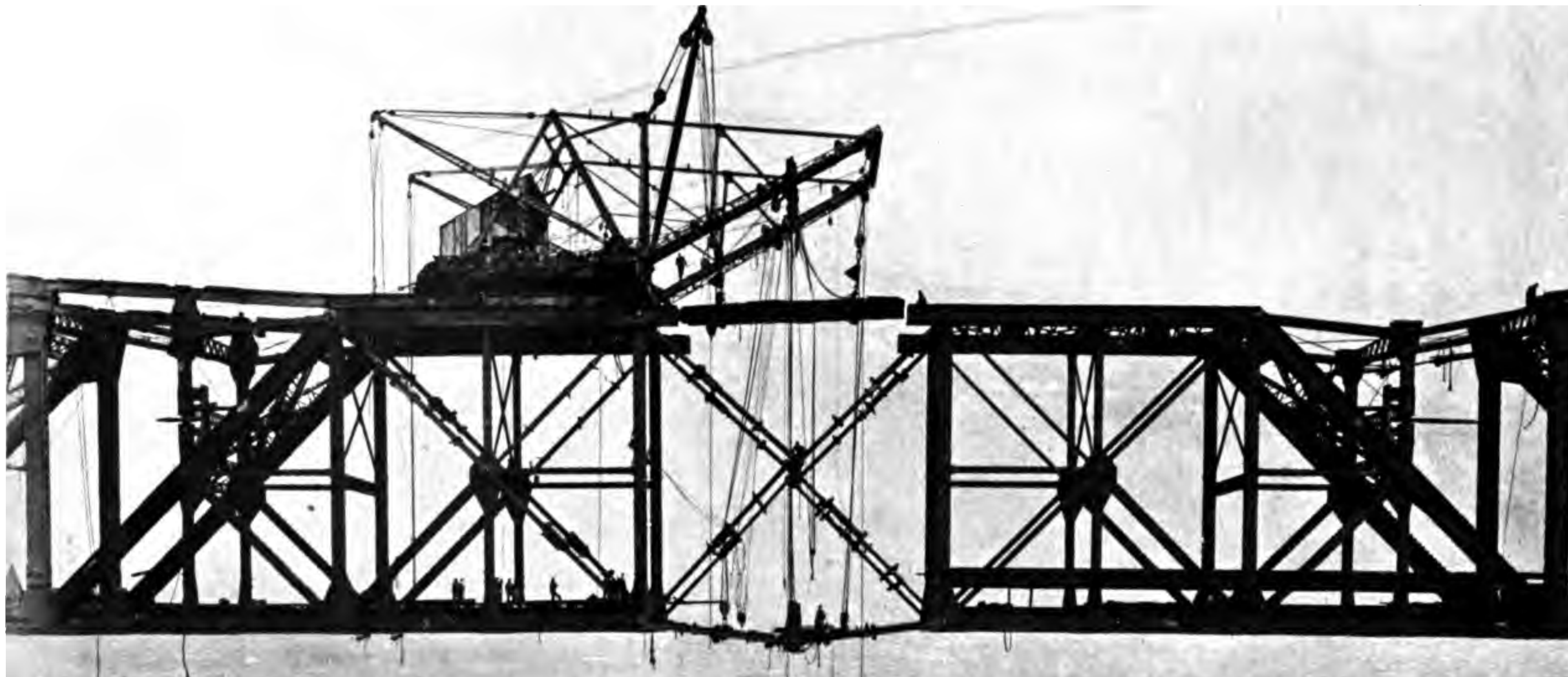


FIG. 69d.

$U_{19}U_{20}$ necessary to permit free expansion and contraction of the main span of the structure under the action of load and temperature.

In making the center connection and transforming the action of the suspended span from that of a cantilever to that of a simple span, in cantilever bridge erection where a stiff bottom chord is used throughout the space between the main piers, great care should be exercised if it becomes necessary to push in the wedges in the bottom chords, as this might overturn the piers.

After the suspended span had been swung the erecting devices were removed. The space occupied by the adjusting device in member $M_{19}U_{19}$ was filled in with a section of part of the proper length to give point U_{19} the desired final position.

The reader is referred to a paper in the Transactions of the Am. Soc. C.E., Vol. LXXIII, by Albert R. Raymer, for full description of the Beaver Bridge.

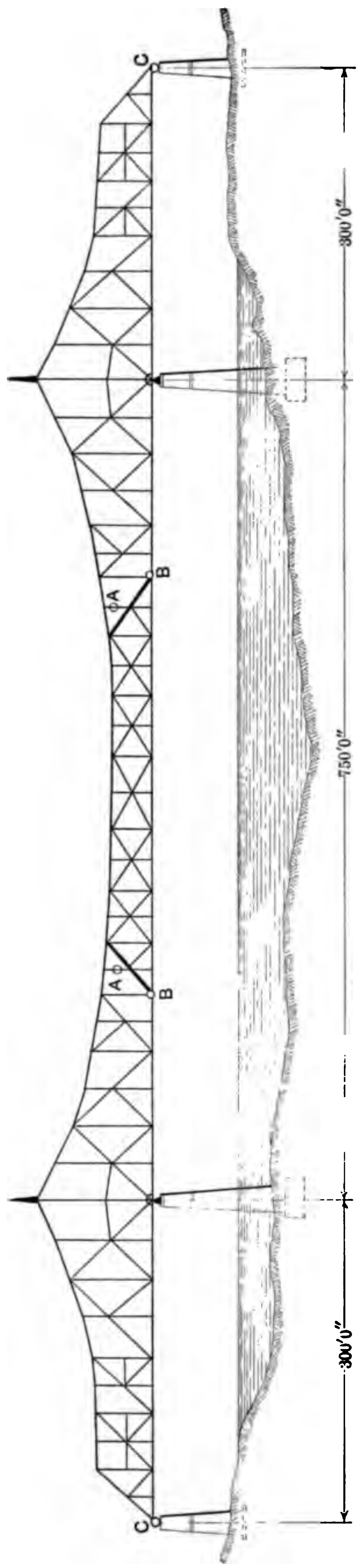
PROBLEM

No. **69a**. Examine the paper of Raymer just referred to; make sketches of the plan, elevation, and a longitudinal cross-section of the anchorage shoe and the permanent adjusting device at this point and write a description of it.

ART. 70. ADJUSTMENTS IN THE ANCHORAGES OF CANTILEVER BRIDGES*

The anchorage for cantilever bridges generally consists of a vertical member composed of eye-bars. These bars must be long enough to provide sufficient masonry, above their attachment at their lower ends, to securely hold the shore ends of the anchor arms under conditions of maximum negative reaction. For the great majority of cantilever bridges there is also a condition of loading giving positive reaction at the anchorage end. There is further a considerable longitudinal motion, to be taken care of, of the shore end of the anchor arms under varying conditions of loading and temperature. In order to show how these varying conditions may be met, the method used for the Ohio River Bridge at Sewickley, Pa., will be illustrated. Let Fig. 70*a* show a general elevation of this bridge, which is for highway use and was built in 1911 by the Fort Pitt Bridge Works of Pittsburgh, Pa. The points marked *A* and *B* on the general elevation show the position of the screw and wedge adjustments used for erection purposes. It is interesting to note in this connection that the adjustments at these points were the same that were used in erecting the Beaver Bridge and which are shown and described in Art. 69. The point marked *C* on the general plan show the location of the anchorage devices. Fig. 70*b* shows the anchorage adjustment devices in consid-

* This article was prepared from data furnished by the Fort Pitt Bridge Works.

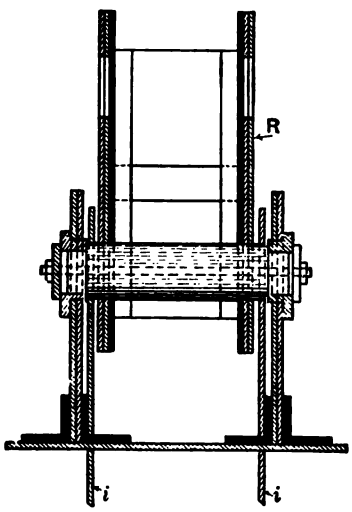


A, Position of Toggle. B, Position of Wedge. C, Position of Anchorage Adjustment.

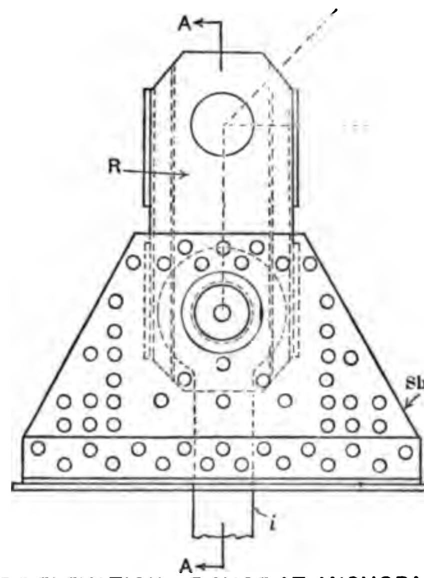
UP STREAM ELEVATION

Fig. 69a.—Bridge over Ohio River, Sewickley, Pa.
Fort Pitt Bridge Works, Pittsburgh, Pa.

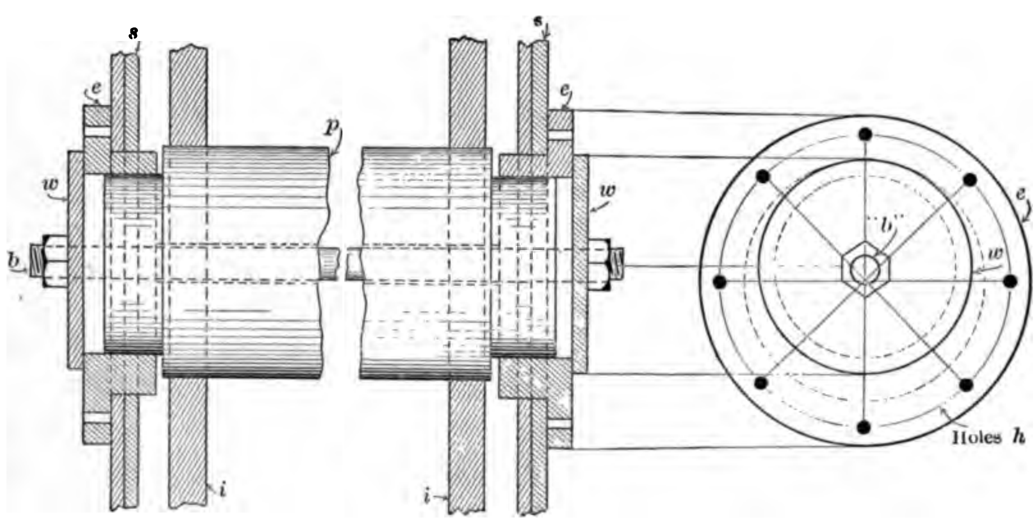
erable detail. The short rockers R which are connected by pins to the bridge at their upper end and to the anchorage at their lower end permit free longitudinal



SECTION A-A



SIDE ELEVATION OF SHOE AT ANCHORAGE



PIN, BOLT, WASHER AND ECCENTRIC IN POSITION
(SHOWING ADJUSTMENT FOR SHOE AT ANCHORAGE)

FIG. 70b.

motion of the end of the anchor arm; they also transmit either tension or compression to their lower pin. Any compression carried by the rockers is carried through the pin p and the shoe s directly to the masonry. As the anchor bars i

are generally long, their elongation under the maximum unit stress would be considerable and as the anchor arm reactions change from plus to minus under the passage of live load there would be hammering of the shoe and a troublesome break in grade line of the roadway at the end of the bridge unless special provision were made to prevent it.

Therefore in general in the erection of a cantilever bridge it is necessary to provide some vertical adjustment at the anchorage shoes to secure their proper bearing on the masonry after the bridge is completed and before it is placed in service. As the erection of the cantilever arm proceeds a negative reaction occurs at the anchorage, causing an uplift at the shoes, and for the bridge of Fig. 70*a* this reaction reached its maximum when the creeper traveler was in the most forward position on the top chord of the suspended span. The ordinary method used in providing for this adjustment is by the means of metal wedges driven under the shoes to maintain the maximum tension, but in the erection of this structure it was provided for by the use of eccentric steel bushings in the pin holes of the shoe.

Referring to Fig. 70*b*, the eccentric bushings *e* are shown in position in pin holes of shoe *s*, and on ends of pin *p*, the pin holes in the shoe having been enlarged and ends of pin turned down for this purpose. The eccentric forms a bushing for the enlarged pin hole and through it the pin strains are transferred to the shoe. The eccentric was made with slight clearances so that it could turn around pin as an axis with a resulting vertical motion to the shoe.

The eccentric consists of a hollow steel casting, with a 2-in. minimum flange in which eight small holes *h* were drilled. It was machine finished, and of such design to be of proper strength and to secure double the amount of vertical adjustment theoretically required. Its first position in the pin hole and on the end of pin was such that the shoe was at its highest elevation with reference to the center of the pin, so that the turning of it in any direction would give a downward motion to the shoe. As the erection of the cantilever arm proceeded, the shoe raised from its bearing on the masonry, and when it reached the highest position above this bearing, the eccentrics on each end of the pin were turned by means of an operating wrench, thus forcing the shoe to its proper bearing on the masonry. Holes were then drilled in the sides of the shoe, through and coincident with the small holes *h*, in the flanges of eccentrics, and tap bolts inserted through these small holes *h* into the newly drilled holes in the sides of the shoe. These tap bolts held eccentrics in place, preventing any further movement around the pin.

The operating wrench for moving the eccentric was made of a steel bar with two prongs on one end to engage two of the small holes in the flange of the eccentric, and long enough so that the strength of two men could easily turn the eccen-

tric. As some time elapsed between the first setting of the shoe and its final adjustment by means of eccentric, the movable parts were well lubricated with a heavy oil in order to prevent oxidation and to reduce friction to a minimum.

PROBLEM

No. 70*a*. Describe the anchorages and adjustment devices used in the Queensborough bridge of New York city.

ART. 71. ADJUSTMENTS FOR ARCHES AND SIMPLE SPANS WHEN ERECTED AS CANTILEVERS*

A. NIAGARA FALLS AND CLIFTON ARCH

The highway and electric railway arch bridge over Niagara Gorge, built in the year 1898 by the Pencoyd Iron Works presented some unique problems of erection. Briefly, each half of the arch was erected as a cantilever, the center connection was made as a three-hinged arch, and the structure was afterward transformed into a two-hinged arch by the artificial introduction in to the center top chord of a compressive stress of known magnitude. This stress was equal to the theoretical stress for this member, computed from the dead load existing at that time, upon the assumption of a two-hinged condition.

Fig. 71*a* shows the arch in completed form.

Fig. 71*b* shows the ties and anchorages which were used for cantilevering the two halves out to the center.

Fig. 71*c* shows a detail of the toggle adjustment used near each anchorage, by means of which the projecting ends of the half arches were lowered to a point where the center bottom chord pins could be driven. The toggles were then slacked off and the structure was thus made to act as a three-hinged arch.

Fig. 71*d* shows a detail of a 250-ton hydraulic jack which was placed at the center of the arch. It had two horizontal plungers located so as to produce pressure upon diaphragms in the top chords. A vertical 30-ton jack was placed under this 250-ton jack, its plunger acting upon the liquid in the latter, the relative areas of the plungers being such that when the small jack developed 30 tons the large one developed 250 tons.

After driving the center bottom chord pin and slacking off the toggles, the hydraulic jacks were operated to produce a stress of 375,000 lbs. in each center

* This article was prepared by Mr. Francis P. Witmer, Des. Engr., American Bridge Co.

top chord, this being the calculated stress at this point, for the two-hinged arch condition, with the dead load then existing.

It was found that the actual movement of the horizontal plungers, after first contact with the chord, checked very closely with the theoretical movement of

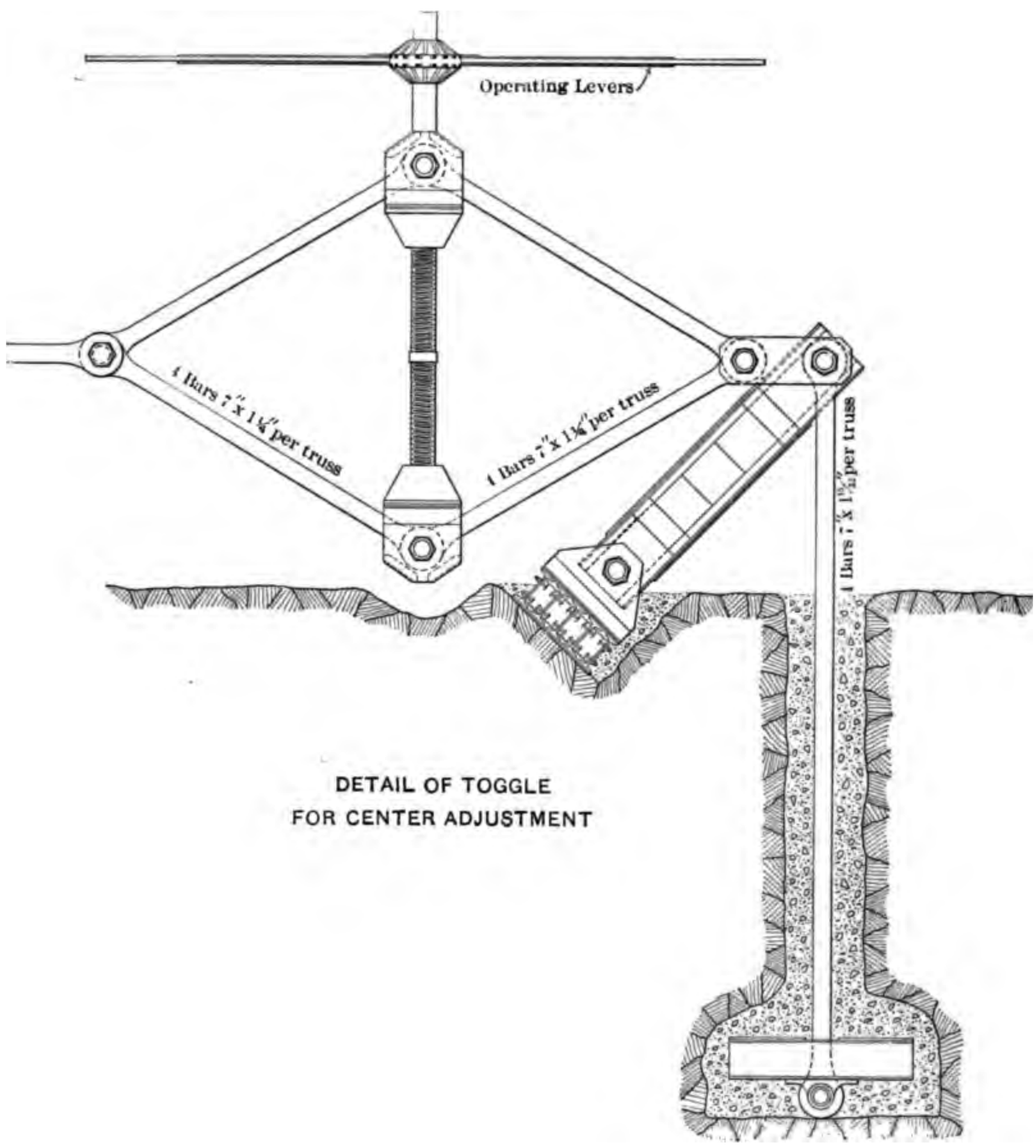


Fig. 71c.

this point as computed from the figured deformations of truss members, due to the change in stress from the three-hinged to the two-hinged condition.

Shim plates were fitted tightly into the opening between the adjacent ends of top chords and the hydraulic pressure was then relieved. Cast-steel blocks were

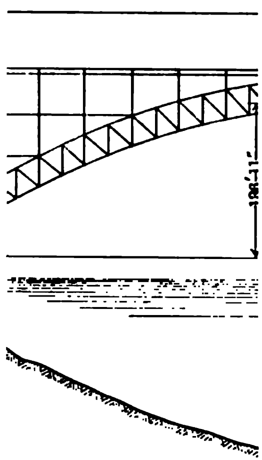


Fig. 71a.—
Built in

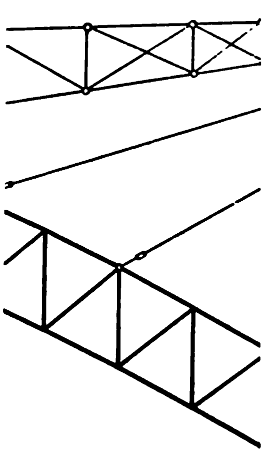
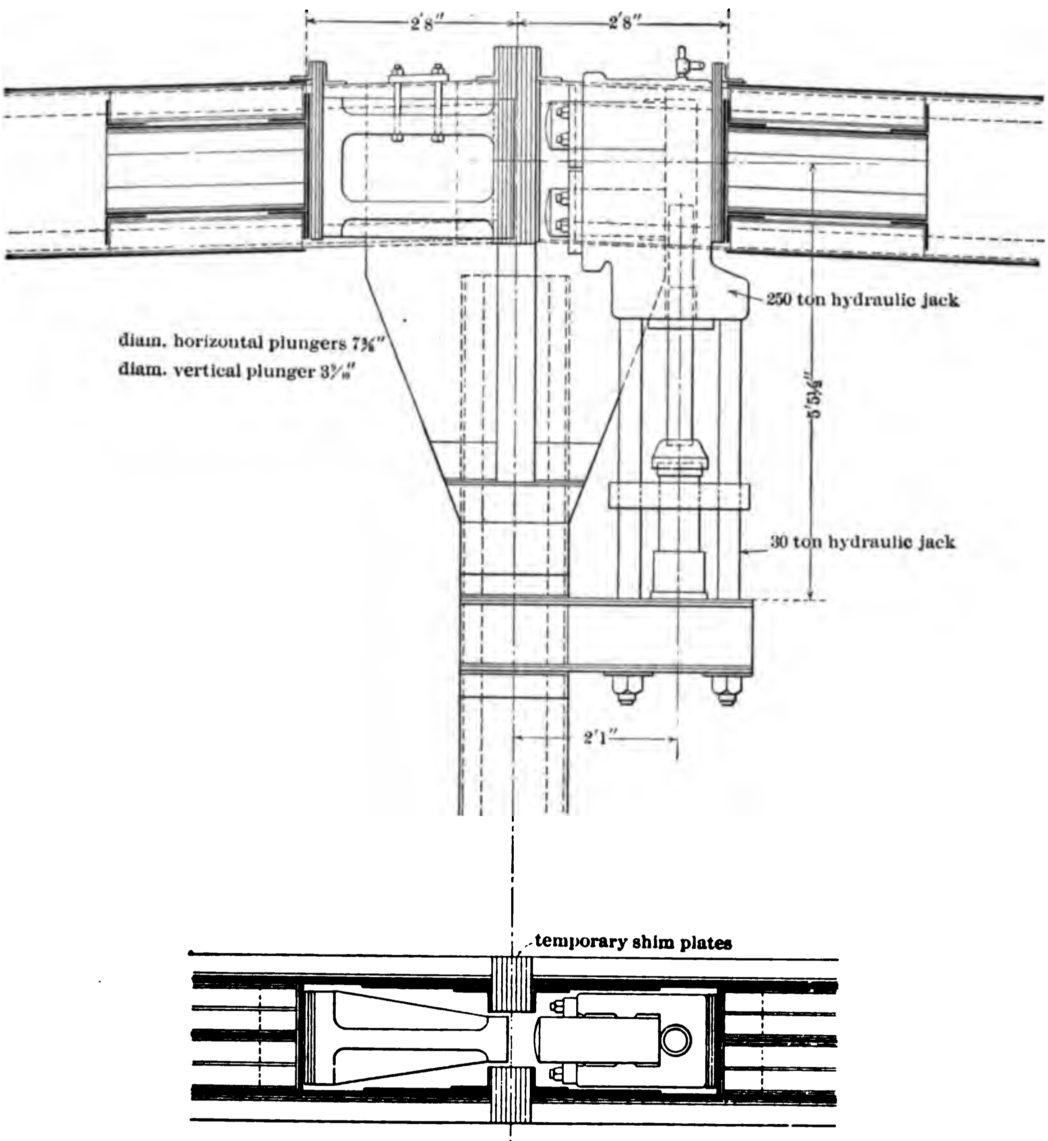


Fig. 71b.—N
Built in

machined to fit exactly into the space occupied by the shims. When these castings were ready the hydraulic pressure was again applied so as to produce once more



DETAILS OF JACKS AT CENTER OF ARCH

Fig. 71*d*

the stress of 375,000 lbs. There was observed no movement of the chord under this second application, indicating that the arch members had held their stresses without change during the interval. The shims were now removed and the castings

inserted in their final position, thus completing the closing and swinging of the span. The erection was then carried to completion by ordinary methods.

B. Benwood Bridge, B. & O. R. R. Co.

In the year 1903 the Baltimore & Ohio Railroad Company renewed spans 11 and 12 of their single-track bridge across the Ohio River at Benwood, W. Va.

The old spans were through Whipple truss spans 342 ft. $4\frac{3}{4}$ ins. and 235 ft. $1\frac{1}{2}$ ins. long, respectively, center to center of end pins. Falsework could be used under span 12, but was not permitted under span 11, so that a cantilever method of erection was adopted for the latter span.

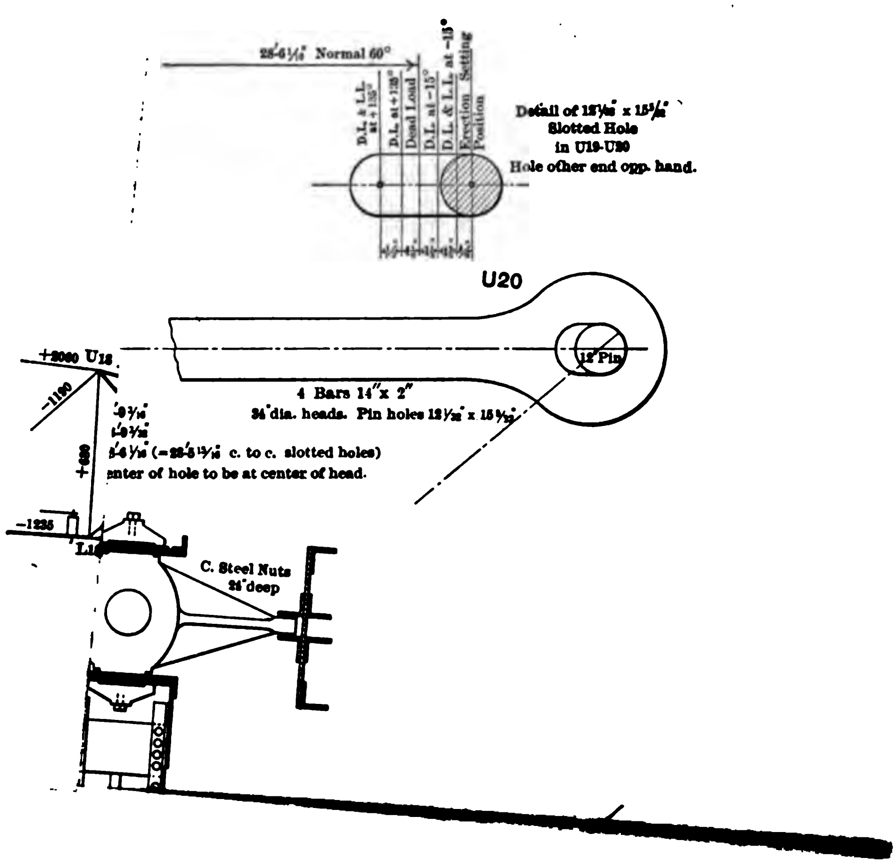
Span 10, a through pin span 207 ft. 0 ins. centre to centre of end pins, had just been renewed, having been completed in September, 1902. It was of an ordinary type, designed without any provision for its use in the future renewal of span 11, and consequently it was not available as an anchor arm for the cantilever erection of that span.

Spans 10, 11 and 12 before renewal are shown in Fig. 71*e*. After renewal, they appeared as in Fig. 71*f*. New span 12 was erected upon falsework, and was designed to serve as an anchor arm for erecting the west half of span 11.

For span 10, two new through pin trusses with parallel chords were supplied for temporary use as an anchor arm in the erection of the east half of span 11. These trusses were erected outside of the existing span 10, and braced to it by transverse struts at each panel point, so arranged as to prevent transverse movement of the erection trusses, but to leave them free to move in a vertical plane. This arrangement was necessary on account of the relative motions of the existing and temporary trusses taking place during the erection of span 11, the existing span 10 being stationary and carrying railroad traffic during this time. The erection trusses were designed to be ultimately used elsewhere in an inverted position, as trusses of a deck pin span.

Fig. 71*g* shows the positions of the various trusses during the erection of span 11, and also indicates the location of the special devices required for the final adjustment of this span.

Fig. 71*h* shows in more detail the arrangement over the pier carrying spans 11 and 12. Owing to the difference in distance between the trusses of these two spans, cross girders were required as shown in the figure, in order to carry the erection stresses across from span 11 to span 12. These cross girders were subsequently used elsewhere to form deck plate girder spans.

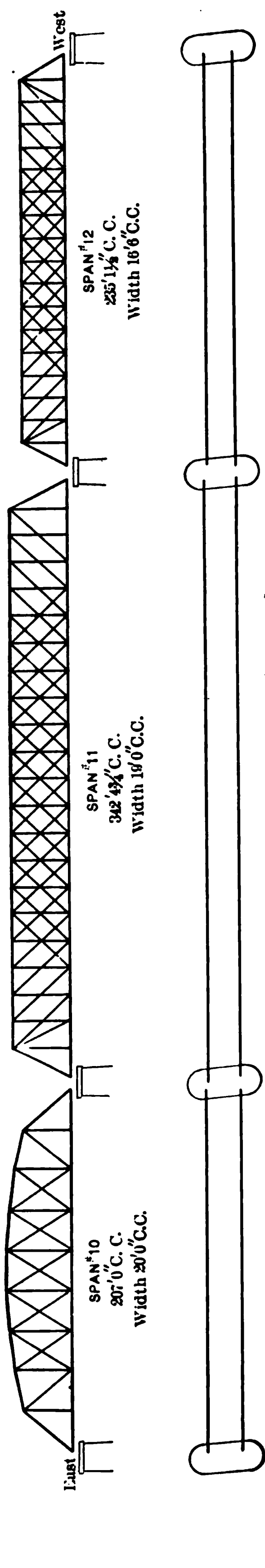


Normal 60°
Dead Load
Erection Setting Position
Detail of
Slotted Hole
Hole other end opp. hand.
U20
Pin
4 Bars 14" x 2"
dia. heads. Pin holes
c. to c. slotted holes)
enter of hole to be at center of head.
C. Steel Nuts
deep

inse
spar

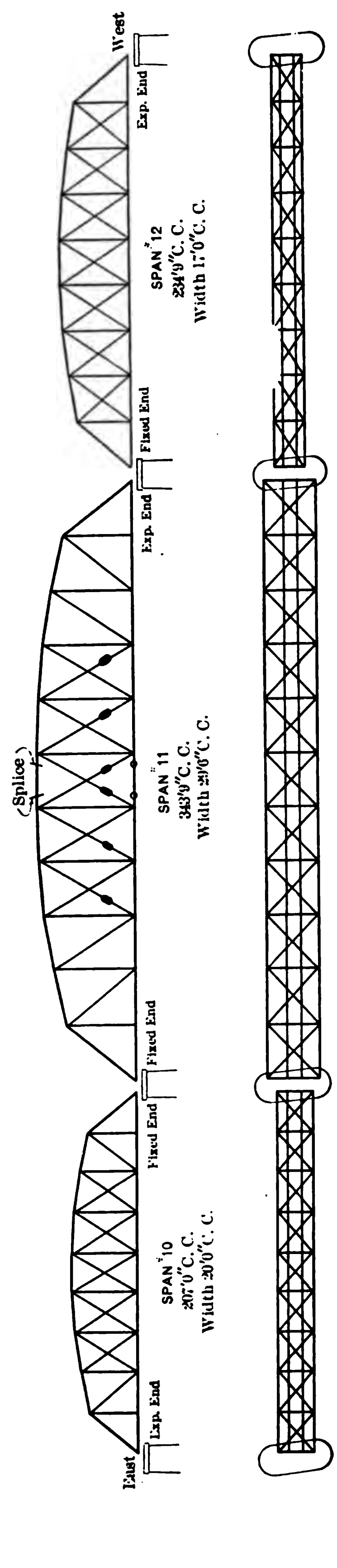
and

1½ in
unde
erect
§
been
type,
and c
of th
S
they
desig
F
tempo
trusse
struts
erectic
ment
porary
being
trusses
trusses
Fi
11, an
adjustr
Fi
11 and
spans,
tion str
used els



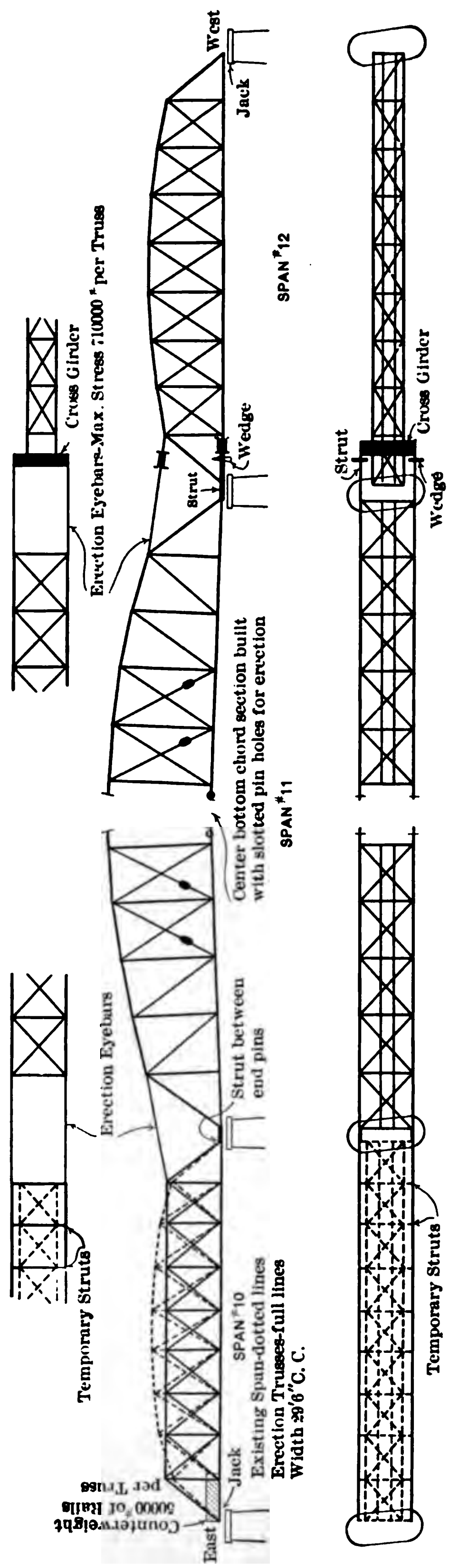
SPANS BEFORE RENEWAL OF #11 AND #12

FIG. 71e.



SPANS AFTER RENEWAL OF #11 AND #12

FIG. 71f.

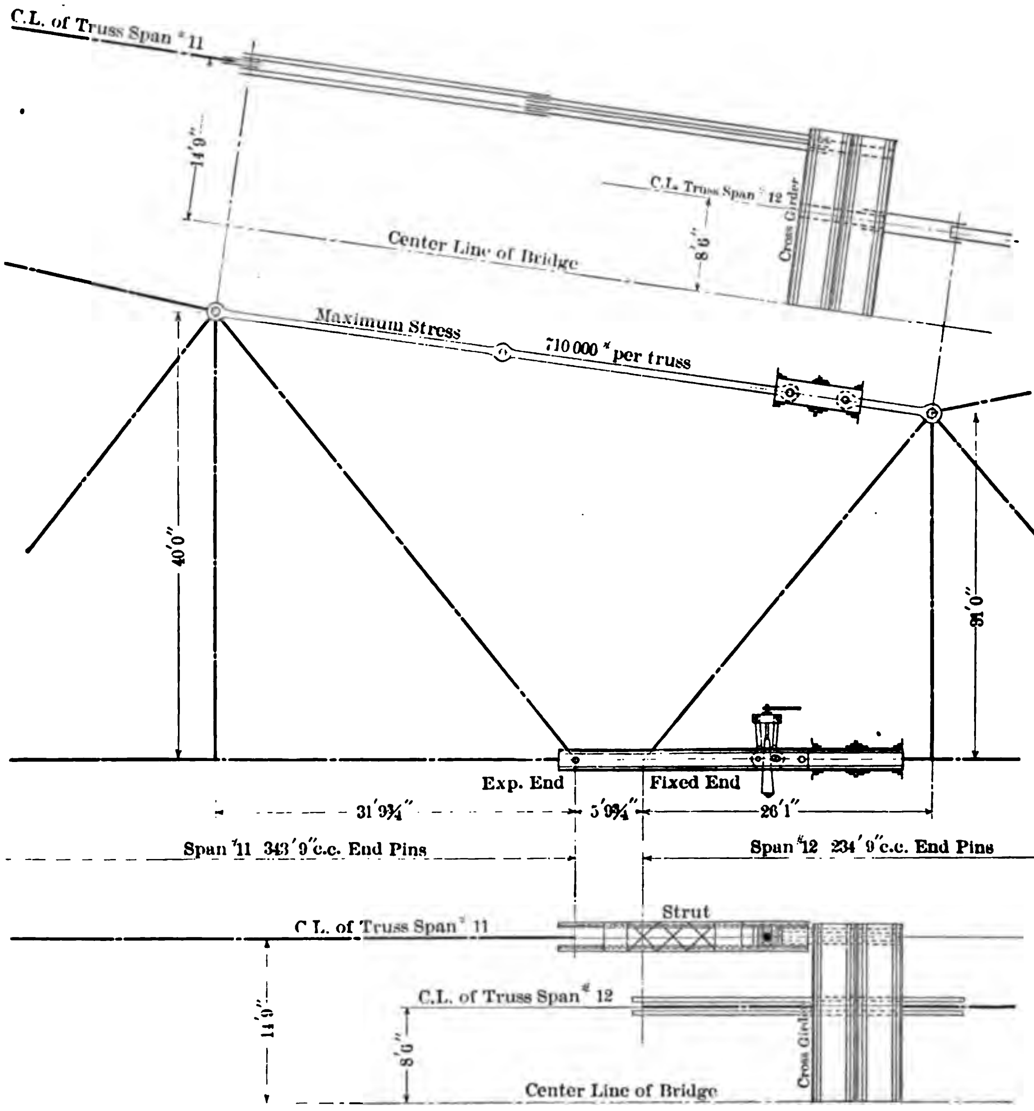


SPANS DURING ERECTION OF #11

Max. lift of E. jack, 33''. Max. lift of W. jack, 19''.
Max. height above level at center of Span No. 11, about 24''.
Max. movement provided for center bottom chord pins, 6½''.

Fig. 71g.—Benwood Bridge, B. & O. R.R. Co.
Rebuilt in 1903 by American Bridge Co., of N. Y.

Fig. 71*i* shows the arrangement over the pier carrying spans 10 and 11. As the erection trusses were placed at the same distance apart as the trusses of span 11, no cross girders were required, the eye-bars in the top chords and the struts in the bottom chords being connected directly to the truss pins of the two spans.



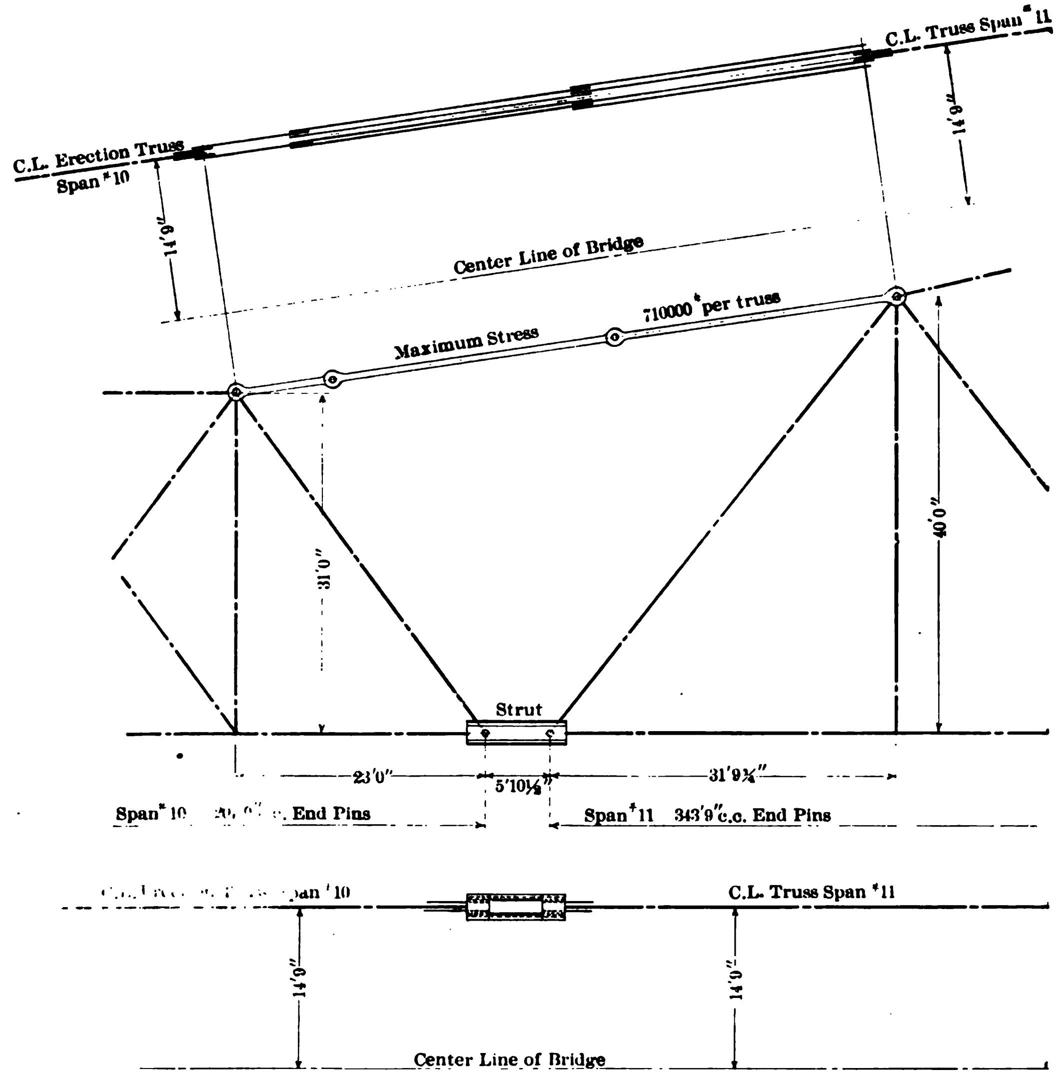
CONNECTION BETWEEN SPANS #11 AND #12

FIG. 71*h*.

The only adjustment devices used consisted of wedges in the bottom chords between spans 11 and 12, and jacks under the west end of trusses of span 12 and under the east end of erection trusses of span 10.

The shoes of span 11 were so located at the outset that when the two halves

of the span were cantilevered out to the center, the lower chords would overlap at least 2 ins. at lowest temperature. This ensured the possibility of driving the closing bottom chord pin. The length of erection eye-bars over piers was made such



CONNECTION BETWEEN SPANS #10 AND #11

FIG. 71*i*.

that the two half spans would be erected high enough at the center to ensure an opening between the ends of adjacent top chords at highest temperature.

When the two halves of the span reached the center, the last bottom chord pins were driven and then the wedges were drawn out, and the east ends of the erection trusses of span 10 were simultaneously jacked up to keep the top chord points of

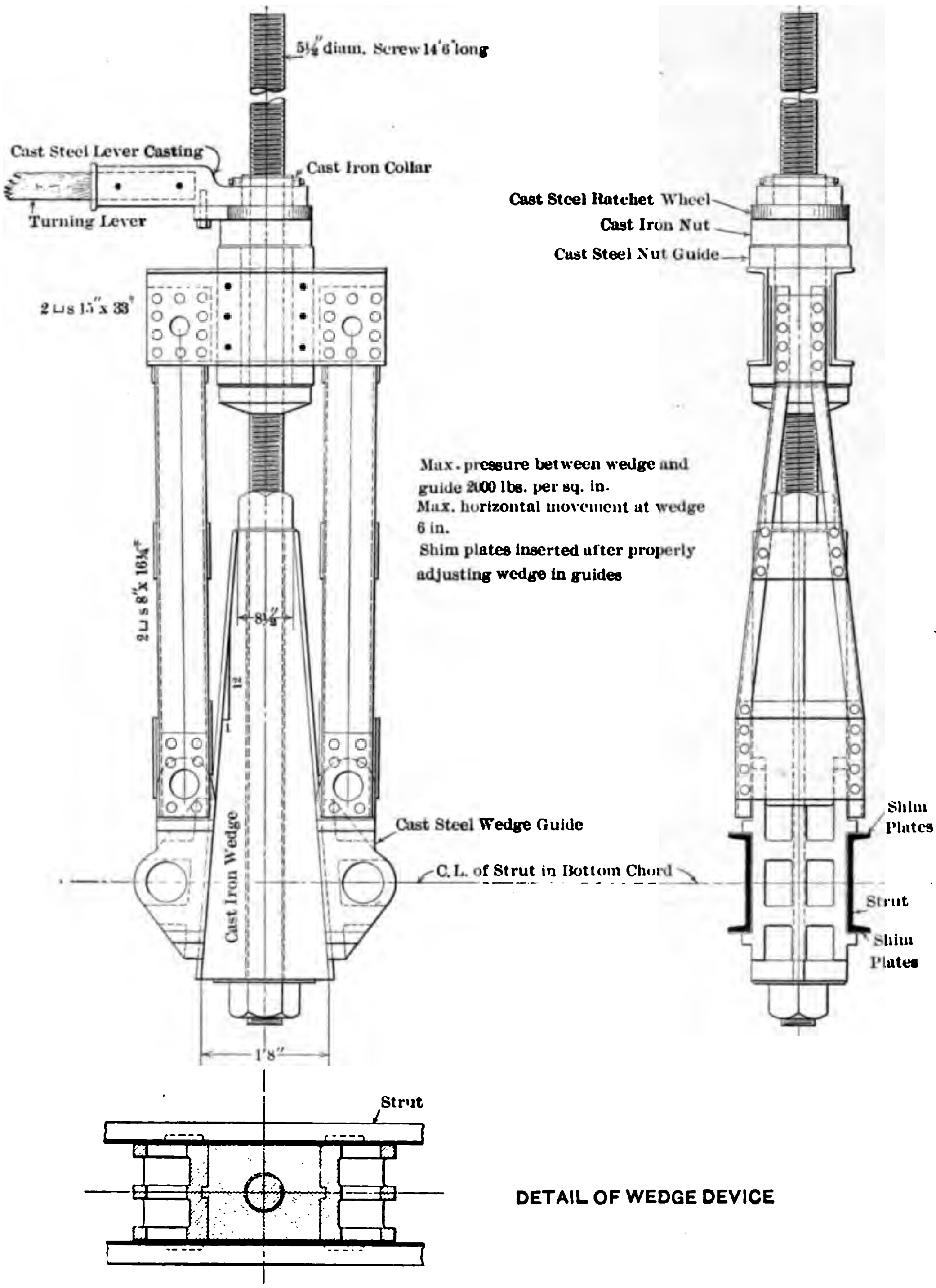

5½" diam. Screw 14'6" long
Cast Steel Lever Casting
Cast Iron Collar
Turning Lever
Cast Steel Ratchet Wheel
Cast Iron Nut
Cast Steel Nut Guide
2 ⊔s 15" x 33"
Max. pressure between wedge and guide 2000 lbs. per sq. in.
Max. horizontal movement at wedge 6 in.
Shim plates inserted after properly adjusting wedge in guides
2 ⊔s 8" X 16¼"
8⅛"
Cast Steel Wedge Guide
Cast Iron Wedge
C. L. of Strut in Bottom Chord
Shim Plates
Strut
Shim Plates
1'8"
Strut
DETAIL OF WEDGE DEVICE

FIG. 71j.

span 11 on either side of the center always at the same level. This operation tended to draw the bottom chords out straight, eliminating the overlap which had been provided and also tended to close up the gap between the adjacent ends of the top chords. The operation was continued until the bottom chords were drawn out straight, and the top chords connected, but with no stress as yet carried across from one half span to the other. The top chords were connected only by bolts at this juncture. During the above operation it was always possible to lower the west end of span 12 if necessary , in order to avoid a compression in the top chords at the center of span 11, should these come into contact before the bottom chords were drawn out tight. Until the latter condition obtained, any such compression would tend to overturn the piers. As a matter of fact, however, this precaution was not found necessary.

After both chords were connected and the bottom chords drawn out straight, the wedges were further withdrawn, with simultaneous jacking up of the east end of the erection trusses of span 10. The center portions of top chords of span 11 were kept truly level by jacking up the west end of span 12 as required. These operations were continued until all erection stresses in eye-bars and struts over the piers were relieved, at which time the cantilever erection stresses in the three spans had been wholly transformed into simple span stresses. Span 11 was thus finally swung. The west end of span 12 was then jacked down to its final position, all erection trusses, eye-bars, struts, wedges, etc., were removed and the erection of the steel floor and bracing was completed.

It should be noted that the wedges were never expected to be driven in during the adjustment of span 11. Their movement was one of withdrawal only, this movement being produced by a long screw which was actuated by a nut, through the medium of a ratchet and turning lever. The details of the wedge device are shown in Fig. 71*j*. The withdrawal of the wedges permitted the west end pins of span 11 to move toward the west, and thus enabled the bottom chords of this span to be drawn out straight, and also to increase in length owing to the change in their stress from compression to tension. The function of the jacks was to keep the two spans properly level at the center and thus to make possible the final connection of the top chords as well as to avoid undesirable stresses in the trusses, and dangerous thrusts on the piers.

CHAPTER XI

MISCELLANEOUS STRUCTURES AND PROBLEMS

ART. 72. SUSPENSION BRIDGES

SUSPENSION bridges are structures for which any extended treatment in this text-book on the general subject of Statically Indeterminate Stresses will not be attempted, as the value of a thorough presentation of the subject to the average structural engineer is somewhat problematical. Suspension bridges are, however, of very great theoretical interest to all civil engineers and for this reason there will be given a method of determining the stresses for a simple case. The method given, however, is capable of extension for the most complex case.

The suspension bridge is most successfully used for long spans and loadings such as occur on highways or streets. It consists essentially of a cable, made of wire of very high strength, and a stiffening truss carrying the floor suspended from the cable. The function of the cable is to carry the entire dead load and the principal part of the live load. The stiffening truss is used to prevent the distortion of the cable under the passage of live load. For light live loads of a continuous or nearly continuous nature a light shallow stiffening truss serves to properly distribute the same and prevent undue deformation of the cable. For very heavy live loads of a concentrated nature the stiffening truss should be deep and heavy. The proper design of a stiffening truss for a suspension bridge then depends on the character and amount of the loading and on the amount of deformation permissible in the structure.

The wire cables for suspension bridges are generally composed of a number of separate wires side by side bound together in a cross-section of circular form, the cable being so made that the wires up to the point of adding the load of the stiffening trusses, have only tensible stress due to their own weight. The stiffening trusses with the floors and bracing are then attached in such a manner as to prevent as far as possible any moment from being developed in the truss.

For small and unimportant suspension bridges cables of wire ropes with twisted strands may be used. The modulus of elasticity and net cross-section of such ropes are much smaller than for the untwisted rope of the same diameter. Such cable may be completed in a shop and erected in place as a whole. For large

cables this method of construction and erection is not possible for reasons which will not be given here.

The *exact* form of curve which a wire cable will assume under full dead load is very difficult of determination, but is very close to that of a parabola.

In order to render the expression for the relation of stress to deformation simple and reduce the labor of determining the lengths of its parts, the cable for the suspension bridge which will be used to illustrate this article will be taken as made of eye-bars.

Let a bridge of the form shown in Fig. 72*a* be assumed for the purpose of illustration.

The verticals V_1 and V_{10} will be assumed to have a pin bearing at both the top and bottom ends.

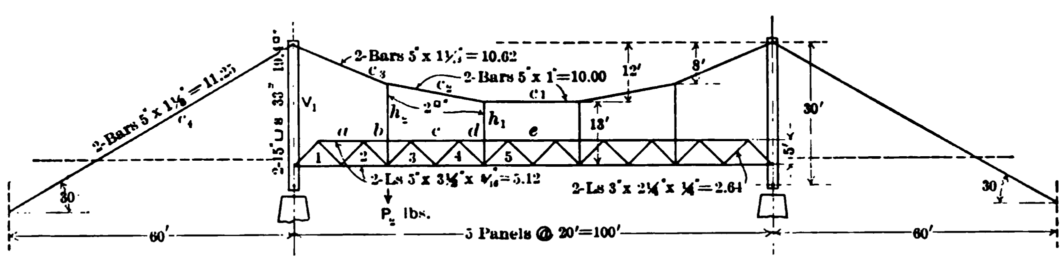


FIG. 72*a*.

The ends of the stiffening truss will be assumed to be supported in such a manner as to permit free horizontal motion and to prevent all vertical motion either up or down.

If the eye-bar cable be considered cut at the middle of c_1, the resulting structure is rendered statically determinate as the cable, the towers V_1 and V_{10}, and the suspenders h_1, h_{10}, h_2 and h_{20} cease to act and the stiffening trusses carry any load which may be on the bridge. It is clear then that the stress in c_1, when the bridge is acting as a whole is that which it is necessary to apply to the cut ends in order to just bring the cut ends together. In order to find this force or stress

Let H = the stress in c_1, when the structure is acting as a whole;

Δ = the relative horizontal motion of the two cut ends of c_1, when c_1 is considered cut at the center, due to any given loading;

d = the relative horizontal motion of the two cut ends of c_1, due to opposite horizontal forces of unity applied to the cut ends.

Then for the structure acting as a whole it must be true that

$$Hd = \Delta, \quad \text{or} \quad H = \frac{\Delta}{d}, \qquad \ldots \ldots \ldots \quad (a)$$

The computations necessary to determine H for any given loading such as a load P_2 at the bottom of h_2 are very quickly made if arranged as shown in Table No. 72a.

TABLE NO. 72a

Member	L = Length in Ft.	A = Area of Section in Sq.ins.	S = Stresses due to P_2 and c_1 cut	T = Stresses due to 1 Pound in c_1	$\frac{TL}{A}$	$\frac{T^2L}{A}=d$	$\frac{STL}{A}=\Delta$	S_H = Stresses due to a Stress of H in c_1	Actual Stresses
a	10.00	5.12	$-1.6P_2$	-0.80	-1.5625	$+\ 1.250$	$+\ 2.500P_2$	$+\ .633P_2$	$-0\ 967P_2$
b	10.00	5.12	$-3.2P_2$	-1.60	-3.1250	$+\ 5.000$	$+\ 10.000P_2$	$+1.265P_2$	$-1.935P_2$
c	10.00	5.12	$-2.8P_2$	-2.00	-3.9063	$+\ 7.813$	$+\ 10.938P_2$	$+1.582P_2$	$-1.218P_2$
d	10.00	5.12	$-2.4P_2$	-2.40	-4.6875	$+\ 11.250$	$+\ 11.250P_2$	$+1.898P_2$	$-0.502P_2$
e	10.00	5.12	$-2.0P_2$	-2.40	-4.6875	$+\ 11.250$	$+\ 9.375P_2$	$+1.898P_2$	$-0.112P_2$
d_1	10.00	5.12	$-1.6P_2$	-2.40	-4.6875	$+\ 11.250$	$+\ 7.500P_2$	$+1.898P_2$	$+0.298P_2$
c_1	10.00	5.12	$-1.2P_2$	-2.00	-3.9063	$+\ 7.812$	$+\ 4.688P_2$	$+1.582P_2$	$+0.382P_2$
b_1	10.00	5.12	$-0.8P_2$	-1.60	-3.1250	$+\ 5.000$	$+\ 2.500P_2$	$+1.265P_2$	$+0.465P_2$
a_1	10.00	5.12	$-0.4P_2$	-0.80	-1.5625	$+\ 1.250$	$+\ 0.625P_2$	$+0.633P_2$	$+0.233P_2$
						61.875	$59.376P_2$		
1	10.00	5.12	$+0.8P_2$	$+0.40$	$+0.7812$	$+\ 0.3125$	$+\ 0.625P_2$	$-0.316P_2$	$+0.484P_2$
2	10.00	5.12	$+2.4P_2$	$+1.20$	$+2.3438$	$+\ 2.8125$	$+\ 5.625P_2$	$-0.948P_2$	$+1.452P_2$
3	10.00	5.12	$+3.0P_2$	$+1.80$	$+3.5156$	$+\ 6.3281$	$+\ 10.547P_2$	$-1.423P_2$	$+1.577P_2$
4	10.00	5.12	$+2.6P_2$	$+2.20$	$+4.2969$	$+\ 9.4532$	$+\ 11.172P_2$	$-1.739P_2$	$+0.861P_2$
5	10.00	5.12	$+2.2P_2$	$+2.40$	$+4.6875$	$+\ 11.2500$	$+\ 10.313P_2$	$-1.898P_2$	$+0.312P_2$
50	10.00	5.12	$+1.8P_2$	$+2.40$	$+4.6875$	$+\ 11.2500$	$+\ 8.437P_2$	$-1.898P_2$	$-0.098P_2$
40	10.00	5.12	$+1.4P_2$	$+2.20$	$+4.2969$	$+\ 9.4532$	$+\ 6.016P_2$	$-1.739P_2$	$-0.339P_2$
30	10.00	5.12	$+1.0P_2$	$+1.80$	$+3.5156$	$+\ 6.3281$	$+\ 3.516P_2$	$-1.423P_2$	$-0.423P_2$
20	10.00	5.12	$+0.6P_2$	$+1.20$	$+2.3438$	$+\ 2.8125$	$+\ 1.406P_2$	$-0.948P_2$	$-0.348P_2$
10	10.00	5.12	$+0.2P_2$	$+0.40$	$+0.7812$	$+\ 0.3125$	$+\ 0.156P_2$	$-0.316P_2$	$-0.116P_2$
						59.3126	$+\ 57.813P_2$		
P_1	14.1421	2.64	$-1.1314P_2$	-0.5657	-3.0304	$+\ 1.714$	$+\ 3.428P_2$	$+0.447P_2$	$-0.684P_2$
T_1	14.1421	2.64	$+1.1314P_2$	$+0.5657$	$+3.0304$	$+\ 1.714$	$+\ 3.428P_2$	$-0.447P_2$	$+0.684P_2$
P_2	14.1421	2.64	$-1.1314P_2$	-0.5657	-3.0304	$+\ 1.714$	$+\ 3.428P_2$	$+0.447P_2$	$-0.684P_2$
T_2	14.1421	2.64	$+1.1314P_2$	$+0.5657$	$+3.0304$	$+\ 1.714$	$+\ 3.428P_2$	$-0.447P_2$	$+0.684P_2$
P_3	14.1421	2.64	$+0.2828P_2$	-0.2828	-1.5149	$+\ 0.428$	$-\ 0.428P_2$	$+0.223P_2$	$-0.060P_2$
T_3	14.1421	2.64	$-0.2828P_2$	$+0.2828$	$+1.5149$	$+\ 0.428$	$-\ 0.428P_2$	$-0.223P_2$	$+0.060P_2$
P_4	14.1421	2.64	$+0.2828P_2$	-0.2828	-1.5149	$+\ 0.428$	$-\ 0.428P_2$	$+0.223P_2$	$-0.060P_2$
T_4	14.1421	2.64	$-0.2828P_2$	$+0.2828$	$+1.5149$	$+\ 0.428$	$-\ 0.428P_2$	$-0.223P_2$	$+0.060P_2$
P_5	14.1421	2.64	$+0.2828P_2$	0.0000	....	....	....	....	$+0.283P_2$
T_5	14.1421	2.64	$-0.2828P_2$	0.0000	....	....	....	....	$-0.283P_2$
T_{50}	14.1421	2.64	$+0.2828P_2$	0.0000	....	....	....	....	$+0.283P_2$
P_{50}	14.1421	2.64	$-0.2828P_2$	0.0000	....	....	....	....	$-0.283P_2$
T_{40}	14.1421	2.64	$+0.2828P_2$	$+0.2828$	$+1.5149$	$+\ 0.428$	$+\ 0.428P_2$	$-0.223P_2$	$+0.060P_2$
P_{40}	14.1421	2.64	$-0.2828P_2$	-0.2828	-1.5149	$+\ 0.428$	$+\ 0.428P_2$	$+0.223P_2$	$-0.060P_2$
T_{30}	14.1421	2.64	$+0.2828P_2$	$+0.2828$	$+1.5149$	$+\ 0.428$	$+\ 0.428P_2$	$-0.223P_2$	$+0.060P_2$
P_{30}	14.1421	2.64	$-0.2828P_2$	-0.2828	-1.5149	$+\ 0.428$	$+\ 0.428P_2$	$+0.223P_2$	$-0.060P_2$
T_{20}	14.1421	2.64	$+0.2828P_2$	$+0.5657$	$+3.0304$	$+\ 1.714$	$+\ 0.857P_2$	$-0.447P_2$	$-0.164P_2$
P_{20}	14.1421	2.64	$-0.2828P_2$	-0.5657	-3.0304	$+\ 1.714$	$+\ 0.857P_2$	$+0.447P_2$	$+0.164P_2$
T_{10}	14.1421	2.64	$+0.2828P_2$	$+0.5657$	$+3.0304$	$+\ 1.714$	$+\ 0.857P_2$	$-0.447P_2$	$-0.164P_2$
P_{10}	14.1421	2.64	$-0.2828P_2$	-0.5657	-3.0304	$+\ 1.714$	$+\ 0.857P_2$	$+0.447P_2$	$+0.164P_2$
						$+\ 17.136$	$+\ 17.140P_2$		
						138.324	$134.329P_2$		

TABLE No. 72*a*—*Continued*

Member.	L = Length in Ft.	A = Area in Sq.ins.	S = Stress due to P_2 and c_1 cut	T = Stresses due to 1 Pound in c_1	$\frac{TL}{A}$	$\frac{T^2L}{A} = d$	S_H = Stress in c_1 due to P_2	Actual Stresses
c_1	20.0000	10.00	0	−1.000	−2.000	+ 2.000	+0.791P_2	+0.791P_2
c_2	20.3961	10.00	0	−1.020	−2.081	+ 2.123	+0.806P_2	+0.806P_2
c_{20}	20.3961	10.00	0	−1.020	−2.081	+ 2.123	+0.806P_2	+0.806P_2
c_3	21.5407	10.62	0	−1.077	−2.184	+ 2.352	+0.852P_2	+0.852P_2
c_{30}	21.5407	10.62	0	−1.077	−2.184	+ 2.352	+0.852P_2	+0.852P_2
c_4	69.2820	11.25	0	−1.155	−7.113	+ 8.217	+0.913P_2	+0.913P_2
c_{40}	69.2820	11.25	0	−1.155	−7.113	+ 8.217	+0.913P_2	+0.913P_2
V_1	30.0000	19.41	0	+0.977	+1.510	+ 1.475	−0.773P_2	−0.773P_2
V_{10}	30.0000	19.41	0	+0.977	+1.510	+ 1.475	−0.773P_2	−0.773P_2
h_1	13.0000	2.00	0	−0.200	−1.300	+ 0.260	+0.158P_2	+0.158P_2
h_{10}	13.0000	2.00	0	−0.200	−1.300	+ 0.260	+0.158P_2	+0.158P_2
h_2	17.0000	2.00	0	−0.200	−1.700	+ 0.340	+0.158P_2	+0.158P_2
h_{20}	17.0000	2.00	0	−0.200	−1.700	+ 0.340	+0.158P_2	+0.158P_2
	$H = \frac{J}{d} = \frac{134.329P_2}{169.858} = .791P_2$					31.534 138.324 169.858		

The headings of the several columns sufficiently explain the table.

For any case of actual design the stresses in every member for a load at every panel point should be determined, just as has been done previously for the one load, and the loading producing maximum stresses in any member thus determined.

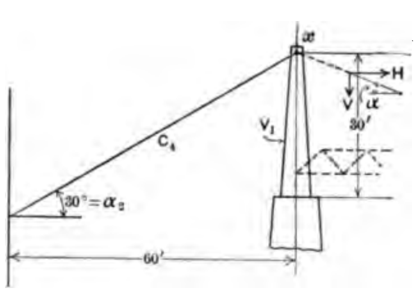


FIG. 72*b*.

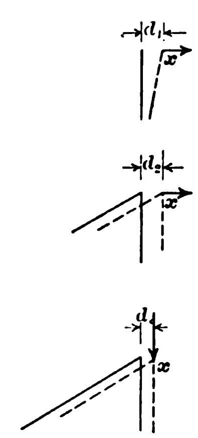


FIG. 72*c*.

In assuming sizes for the preliminary design for a suspension bridge, make the cable of sufficient size to carry the full live and dead load at the required unit stress; make the chords of the suspended span for a moment at the center due to a live load of $\frac{1}{4}$ that specified, making the top and bottom chords alike and of the same section throughout; and make the web members throughout the truss for a shear equal to that at the end of the span due to a live load $\frac{1}{4}$ of that specified.

Where the vertical members V_1 and V_{10} are fixed to the masonry by sufficient anchorage their effect in reducing the horizontal component of cable stress transferred from the center to the side spans of the cable should be considered.

That is for a construction as shown in Fig. 72*b* the horizontal component of the cable tension in the center span is resisted by the stiffness of the post V_1 in bending and by the horizontal component of stress in c_4 jointly.

In order to find the horizontal component of stress in c_4 the following nomenclature will be defined:

Let H and V = the horizontal and vertical components respectively of the cable stress for the cable on the side of the tower toward the center of the bridge;

H_2 and V_2 = the same quantities for the cable on the shore side of the tower;

H_1 = the horizontal force caused by the tower post due to its stiffness as a cantilever beam;

V_1 = the vertical stress in the tower post = its direct stress;

d_1 = the horizontal deflection of x of the tower post produced by the flexural stresses due to a horizontal force of unity at x;

d_2 = the horizontal deflection of x of the framework due to a horizontal force of unity at x;

d_3 = the horizontal deflection of x of the framework due to a vertical load of unity at x.

The following equations may now be written:

$$H_1+H_2=H, \quad \dots \dots \dots \quad (1)$$

$$V+V_2=V_1=H\tan\alpha+H_2\tan\alpha_2, \quad \dots \dots \dots \quad (2)$$

$$H_1d_1=H_2d_2+V_1d_3=H_2d_2+(H\tan\alpha+H_2\tan\alpha_2)d_3, \quad \dots \quad (3)$$

From (3)

$$H_1=\frac{d_2}{d_1}.H_2+\frac{d_3}{d_1}H\tan\alpha+\frac{d_3}{d_1}H_2\tan\alpha_2.$$

Substituting this value of H_1 in (1) and solving

$$H_2=H-\frac{d_2}{d_1}.H_2-\frac{d_3}{d_1}H\tan\alpha-\frac{d_3}{d_1}H_2\tan\alpha_2;$$

$$H_2\left(1+\frac{d_2}{d_1}+\frac{d_3}{d_1}\tan\alpha_2\right)=H\left(1-\frac{d_3}{d_1}\tan\alpha\right);$$

from which there is given a value for H_2 in terms of known quantities,

$$H_2 = H\left(\frac{d_1 - d_3 \tan \alpha}{d_1 + d_2 + d_3 \tan \alpha}\right).$$

In this determination the horizontal deflection of x of the tower post due to the bending produced by V_1 has been neglected, as it would in any case be an extremely small quantity.

If the cable in the shore spans supports stiffening trusses, their effect should be included in determining d_1.

The amount of the horizontal component of the cable tension for any given change in temperature may be determined from formula (a) of this article, the value of Δ to be used being that due to the changes in length of the various members for the change in temperature above or below the normal.

The previous method of determining the stresses in a suspension bridge is based on the assumption of very slight deformations of the structure under the loadings producing maximum stresses.

As a matter of fact in such bridges the deformations are quite large and for very important structures the actual form of the structure under the loadings producing maximum stresses in certain representative members should be ascertained and the stress computations revised to correspond with the true shape under the special loading.

PROBLEMS

No. **72*a*.** Compute the stresses in all the members of the suspension bridge of Fig. 72*a* for a load P_1 at the foot of the suspender h_1.

No. **72*b*.** By means of the tables No. 72*a* and the one prepared for the solution of the previous problem, compute the maximum and minimum stresses in all the members of the bridge of Fig. 72*a* for a uniform dead load of 800 per linear foot of bridge or 8000 lbs. at the foot of each suspender, and a moving live load of 20,000 lbs. at the foot of each suspender.

No. **72*c*.** Compute the stresses in all the members of the suspension bridge of Fig. 72*a* due to rise in temperature of 70° F.

ART. 73. STRESSES IN A VIADUCT BENT HAVING DOUBLE INTERSECTION BRACING

The structure assumed to illustrate the method to be used in computing the stresses in the members of a viaduct bent with double intersection bracing is that of Fig. 73*a*. The adjacent spans supported by the bent will be taken as 30 ft. in length. The wind loading consists of 9000 lbs. on the train, 6000 lbs. on the girders and track, and 1500 lbs. on the upper half of the bent taken as applied at the top of the bent. This gives a total horizontal wind force of 16,500. The

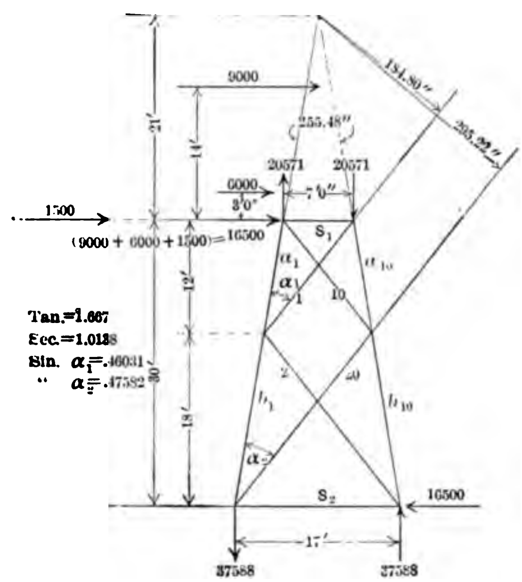


FIG. 73*a*.

forces of 9000 and 6000 lbs. produce a couple acting at the top of the bent, each force of which equals 20,571 lbs. The combined effect of all the wind forces may be represented by a horizontal force of 16,500 lbs. and a couple, each force of which equals 20,571 lbs.

L.L. concentration	=	86,250 lbs.		(Cooper's E_{40})
D.L. "	=	14,250	"	$(30\times400+15\times150)$
Total	=	100,500	"	

The nomenclature, and table follow:

The member S_1 will be considered as the superfluous bar.

Let T = stresses due to a stress of unity in S_1;

A = areas of members;

S_P = stresses due to two concentrated vertical loads of 100,500 (S_1 considered cut at the center);

S_W = stresses due to wind loads (S_1 considered cut at tne center)·

S_x = true stresses due to $ll+dl$;

S_y = true stresses due to wind loads.

TABLE No. 73*a*

Members.	Length. Ins.	A Sq. ins.	T	S_P	$\frac{TL}{A}$	$\frac{TLS_P}{A\times 10000}$	$\frac{T^2L}{A}$	S_x	S_W	$\frac{TLS_W}{A\times 10000}$	S_y
S_1	84.00	3.14	+1.000	− 00000	+26.75	000.00	+ 26.75	− 14400	00000	− 00000	− 8100
1	180.00	3.14	−1.364	− 22844	−78.19	+178.62	+106.65	− 3200	− 4676	+ 36.56	+ 6400
10	180.00	3.14	−1.364	− 22844	−78.19	+178.62	+106.65	− 3200	−17824	+139.37	− 6700
2	273.64	3.14	+0.854	+ 14298	+74.43	+106.42	+ 63.56	+ 2000	+ 2927	+ 21.79	− 4000
20	273.64	3.14	+0.854	+ 14298	+74.43	+106.42	+ 63.56	+ 2000	+11157	+ 83.04	+ 4200
S_2	204.00	4.12	−0.412	+ 9853	−20.40	− 20.10	+ 8.40	+ 15800	−11647	+ 23.76	− 8300
a_1	145.99	14.11	+1.106	− 83362	+11.44	− 95.37	+ 12.65	− 99300	+35311	+ 40.40	+26300
a_{10}	145.99	14.11	+1.106	− 83362	+11.44	− 95.37	+ 12.65	− 99300	−17063	− 19.52	−26000
b_1	218.98	14.11	−0.683	−113329	−10.60	+120.13	+ 7.24	−103500	+29177	− 30.93	+34700
b_{10}	218.98	14.11	−0.683	−113329	−10.60	+120.13	+ 7.24	−103500	−40449	+ 42.88	−34900
						+599.50	+415.35			337.35	

The preceding tabular result of the computations should require no comment.

ART. 74. PROBLEMS

The following problems should be solved by the student.

No. **74*a*.** Fig. 74 *a* represents the cross-section of a single-track through Pratt truss railway bridge. The moment of inertia of the top strut is 40,000 ins. fourth, of the vertical post 365 ins.

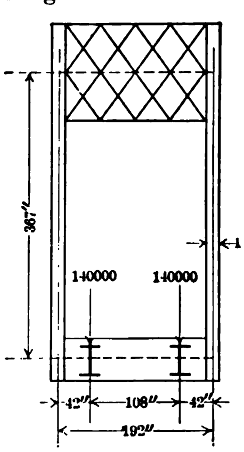


FIG. 74*a*.

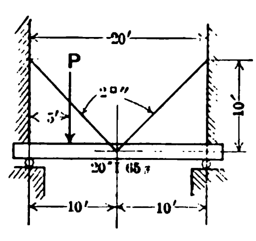


FIG. 74*b*.

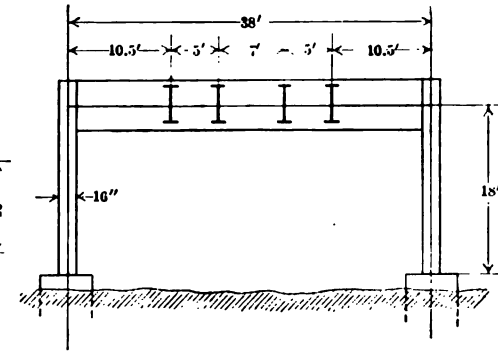


FIG. 74*c*.

fourth, and of the floor beam 12,500 ins. fourth. What are the maximum fiber stresses in the vertical post due to the bending induced in the members of the cross-section by the two con-

centrated loads on the floor beams. The top strut and floor beam will be considered connected to the posts in a manner sufficient to develop the moments at the connections. The material of the structure is steel.

No. **74*b*.** Make a sketch showing the distribution of the moments for the cross-section of the bridge of the previous problem.

No. **74*c*.** Compute the stresses in the rods and beam of the structure of Fig. 74*b*, all in terms of the load P. Material, steel.

No. **74*d*.** For a structure similar to that of Fig. 74*b* determine the size of rods to be used so that the unit stresses in the rods and the 20-inch I 65 lbs. will be equal.

No. **74*e*.** Let Fig. 74*c* represent the transverse elevation of a bent carrying a double-track elevated railway. Let the concentrated loads at each of the longitudinal girder connections be 100,000 lbs., the moment of inertia of the transverse girder 40,000 ins. to the fourth and the area 80 sq.ins. and the depth 75 ins.; the moment of inertia of the posts 800 ins. to the fourth, the area 20 sq.ins. and their width 16 ins.; compute the maximum bending moment and corresponding fiber stresses in the posts under the assumption that the posts are fixed at the bottom ends.

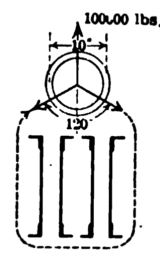


FIG. 74*d*.

No. **74*f*.** Do the vertical posts of the previous problem need any connection to the masonry pedestals to render them theoretically fixed?

No. **74*g*.** Fig. 74*d* shows a sling around one end of a heavy compression member. A ring 10 ins. in diameter connects the two ends of the sling and the lifting attachment. Find the proper size of round wrought-iron bar out of which to make the ring.

INDEX